# CALCULUS
## With Applications
HAYES

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### AN INTRODUCTION TO THE MATHEMATICAL TREATMENT OF SCIENCE

BY

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# PREFACE.

This little book has been written for two classes of persons: those who wish, for purposes of culture, to know, in as simple and direct a way as possible, what the calculus is and what it is for; and students primarily engaged in work in chemistry, astronomy, economics, etc., who have not time or inclination to take long courses in mathematics, yet who would like "to know how to use a tool as fine as the calculus."

The 'pure' mathematician will note the omission of various subjects that are important from his point of view; but for him there are admirable and lengthy treatises on pure calculus. Also the student whose experience has led him to conceive of mathematical study as the doing of interminable lists of exercises, will be surprised and, possibly, disappointed. This book is a reading lesson in applied mathematics. Fancy exercises have been avoided. The examples are, for the most part, real problems from mechanics and astronomy. This plan has been pursued in the conviction that such problems are just as good as make-believe ones for purposes of discipline, and a good deal better for purposes of knowledge. The time-honored method of presenting calculus is much as if travelers should be stopped and made to pound stone on the high-way, so that they never get anywhere or even know what the road is for. The following pages are a protest against the conventional method; for I am wholly in sympathy with a remark made by Professor Lester F. Ward, in his *Outlines of Sociology:* "There is no more vicious educational practice, and scarcely any more common one, than that of keeping the student in the dark as to the end and purpose of his work. It breeds indifference, discouragement, and despair."

A chapter on analytic geometry has been introduced, in the hope that teachers will try the plan of presenting the elements of the calculus and of analytic geometry together. There is no good reason either for keeping them distinct or for presenting analytic geometry first.

To three works I have to express my deep obligation. The spirit manifest in them has been my chief encouragement in preparing this book. I refer to Greenhill's *Differential and Integral Calculus*, Perry's *Calculus for Engineers*, and Nernst and Schönflies' *Einführung in die mathematische Behandlung der Naturwissenschaften.*

We have in these works, let us hope, an indication of the rôle which the calculus is to play in schemes for liberal and scientific education in the not far distant future.

ELLEN HAYES.

Wellesley College,
September, 1900.

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*"Perhaps we should all know how to use a tool as fine as the calculus."* — J. McKeen Cattell.

*"A man learns to use the calculus as he learns to use the chisel or the file on actual concrete bits of work."* — John Perry.

# CALCULUS.

## CHAPTER I.

### DIFFERENTIATION AND INTEGRATION.

**1.** In the experiences of every-day life and in the operations of science, people are continually dealing with things which keep changing in quantity, and with things so connected that a change in one of them is followed by a change in another. For instance, we know that work varies in amount, and that the amount done by workmen depends on the time. We find crops now abundant and now scanty; and, other things being equal, the crops are seen to vary with the amount of fertilizer used. We observe that the height of the mercury in a thermometer changes with the temperature.

In these examples and all similar ones there is a relation of cause and effect, or at least a relation of antecedent and consequent; and we say that a quantitative change in the cause is accompanied by a quantitative change in the effect.

It is part of the business of science not only to discover relations of cause and effect, but also to try to express these relations with precision. When a relation of cause and effect can be stated with exactness, the language of mathematics is the best one to use, because it gives compact and unambiguous expressions, and because a further examination of the relation may then be conducted in that language and results easily reached which could be arrived at only with much difficulty, if at all, in any ordinary language. For example, the expansive force of air was a property observed by Guericke (1602–1686), but Boyle (1627–1691) discovered that the volume varies inversely as the pressure. That is, if $v$ represents the volume of a given quantity of air and $p$ its pressure on unit area of the containing vessel, $v \propto \frac{1}{p}$, and $pv =$ a constant. We shall see later how we may learn more about this law by using the equation $pv = c$. Experience in balancing bodies of equal or unequal weights no doubt furnished ancient craftsmen with some vague notions regarding equilibrium; but Archimedes (287(?)–212 B.C.), from a few assumptions, concluded that two bodies suspended from a bar are in equilibrium when their distances from the point of support of the bar are inversely proportional to their weights. That is, if $l$, $l'$ are the distances of the bodies whose weights are $w$, $w'$, respectively, $w : w' :: l' : l$. The principle of the lever, as thus stated by Archimedes, was later fully established. To illustrate further, from earliest times men must have noticed that unsupported bodies fall to the ground; but after the investigations made by Newton (1642–1727), it was possible to state the law of gravitation with mathematical accuracy: The mutual attraction (stress) between any two bodies varies directly as the product of their masses and inversely as the square of their distance from each other.

Thus, if $F$ is the whole attraction between the earth and the moon, for instance, $M$ the mass of the earth, $m$ the mass of the moon, and $r$ the distance between them, $F = c\frac{Mm}{r^2}$. These examples go to show that when a precise quantitative statement can be made in science, mathematics, with its unambiguous symbolic shorthand, offers the most economical way of making it.

**2.** Two modes of quantitative change or variation present themselves. As an illustration of the first, the number of roses in a handful may be varied by adding one and another and another, until the number has changed from $a$ to $b$; or we may add several at a time until the number has changed from $a$ to $b$. But we cannot do less than add one whole rose at a time; for, in this case, the variation element is a whole unit, that is, a whole rose, and not any fraction of it. Again, we may measure a day with a minute as a unit of measure, and say that a day contains 1440 minutes; but this is only an artificial convenience. Time does not increase a minute "at a time," or even a second at a time, but by elements of time which are immeasurably small fractions of a second. This is the second mode of variation: a quantitative change not by jumps or finite amounts, but by indefinitely small amounts.

**3.** By the term **variable** we mean a quantity which changes in the second manner above described. We use it in speaking of such things as volume, pressure, distance, etc., when they are conceived as being in a state of continuous variation. The term **function** is applied to the quantity which necessarily changes because of a change in a variable with which it is connected. For example, the pressure of steam on the piston of the cylinder is a function of the volume of the steam. The attraction which the earth exerts on the moon is a function of the distance of the moon from the earth.

If the symbol $x$ stands for the variable and $y$ for the function, we briefly express the fact of their connection by the general statement $y = f(x)$. The precise nature of the connection is shown by specializing $f(x)$. For example, if $f(x)$ is $\log x$, we have the particular statement $y = \log x$.

When we need to distinguish one function from another, we use such forms as $F(x)$, $\phi(x)$, $u$, $v$, etc. The nature of these conventional symbols should be carefully noticed. The parenthesis merely serves to separate the quantity symbol $x$ from the other symbols $f$, $F$, etc., which are not quantity symbols and hence not factors. $f(x)$ is only algebraic shorthand for the expression "a function of the varying quantity $x$."

Functions are classified as **algebraic** and **transcendental.** An algebraic function is defined as one which implies only a finite number of the algebraic operations, addition, subtraction, multiplication, division, involution, and evolution.

The trigonometric functions $\sin x$, $\cos x$, $\tan x$, etc., are transcendental; so also are $e^x$, $\log x$, $\sin^{-1} x$, etc.

**4.** Let us now suppose a change in pressure, or distance, or time, or whatever quantity we are dealing with under the symbol $x$. Let $\delta x$ stand for the amount of change. Then the new quantity is represented by

$x + \delta x$, and $f(x)$ becomes $f(x + \delta x)$. Subtracting the former value from the latter, we have $f(x+\delta x)-f(x)$, the amount of change in the function occasioned by the change in the variable. It may be represented by $\delta y$ if $y=f(x)$. Then $\frac{f(x+\delta x)-f(x)}{\delta x}$, or its equal $\frac{\delta y}{\delta x}$, is the ratio of the increment of the function to that of the variable. Let $\delta x$ now be supposed to become smaller and smaller until we cannot tell the difference between it and zero. We say it "has zero for its limit," or it "diminishes without limit," and to show that this supposition has been made we use the symbol $dx$ in place of $\delta x$, and also use $dy$ in place of $\delta y$ for the indefinitely small change in the function. $dx$ is called the **differential** of $x$, and $dy$ the differential of $y$. The ratio $\frac{dy}{dx}$ is called the first differential coefficient of $f(x)$ with respect to $x$, or briefly, the **derivative**.

We shall find that the ratio $\frac{dy}{dx}$ is itself, in general, some function of $x$; hence it is often written $f'(x)$. The symbols $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$, $f'(x)$ all mean the same thing. It is important to notice that although $dy$ and $dx$, the individual terms of the ratio $\frac{dy}{dx}$, are indefinitely small, the ratio itself is usually finite.

This $dx$ of the mathematician is suggestive of the "atom" of the chemist, the "particle" of the physicist, and even the "cell" of the biologist. It is the ultimate element of that with which the mathematician deals, and always implies one property of the quantity symbolized by $x$; namely, its continuous variation.

**5.** To illustrate the nature of $\frac{dy}{dx}$ let $f(x) = \pi x^2$, the area of a circle. Suppose this circular area to be cut out of a thin sheet of metal and to have heat supplied to it in such a way as to cause it to expand, but to remain circular. Let the radius increase by the amount $\delta x$; then

$$y + \delta y = f(x + \delta x) = \pi(x + \delta x)^2$$
$$= \pi x^2 + 2\pi x\delta x + \pi(\delta x)^2,$$

and $\delta y = f(x + \delta x) - f(x) = 2\pi x\delta x + \pi(\delta x)^2$;

hence $\frac{\delta y}{\delta x} = 2\pi x + \pi\delta x = \pi(2x + \delta x)$.

Now, if $\delta x$ be made indefinitely small, the limit of $2x + \delta x$ is $2x$, and therefore

$$\frac{dy}{dx} = 2\pi x.$$

This means that if the quantity of heat used is so small that the increase in the length of the radius is indefinitely small, the ratio of the increment of the area to the increment of the radius is equal to the circumference of the circle, a result which might have been guessed beforehand if we had reflected that the growth in area is a belt only $dx$ wide around the circle, and $dx$ is "next to nothing."

As another illustration, suppose we take the equation $y = \frac{c}{x}$, which states the law concerning the mutual dependence of the volume and pressure of a gas, $x$ representing volume and $y$ representing pressure. Let us look at pressure as a function of volume; then

$$y = f(x) = \frac{c}{x},$$

and $$y + \delta y = f(x + \delta x) = \frac{c}{x + \delta x};$$

therefore $$\delta y = \frac{c}{x + \delta x} - \frac{c}{x} = -\frac{c\delta x}{x(x + \delta x)},$$

and $$\frac{\delta y}{\delta x} = -\frac{c}{x(x + \delta x)};$$

hence $$\frac{dy}{dx} = -\frac{c}{x^2}.$$

In other words, the ratio of the increment of the pressure to the increment of the volume is inversely as the square of the volume. The minus sign means that when the volume takes an increment the increment of the pressure is negative; that is, the pressure-increment is really a *decrement*. This agrees with what is said by the equation itself; namely, that the pressure decreases as the volume increases, and *vice versa*.

If the student will follow the thought in a few concrete examples like the two just given, he will gain a better insight into the nature and purpose of the calculus than he can acquire from the mechanical working of a great number of meaningless exercises.

**6.** The importance of the ratio $\frac{dy}{dx}$ is soon realized by the student of the mathematical side of any of the sciences which admit of mathematical treatment, such as astronomy, thermodynamics, electricity, chemistry, economics, etc. Granting its importance, we should know how to find, by the most direct method, the derivative of any ordinary algebraic or transcendental function; and, what is even more essential, we should be able to perform the reverse operation; that is, having $\frac{dy}{dx}=f'(x)$ to find $f(x)$. In the following articles, theorems are established by means of which derivatives may be directly written, so that we need not take any intermediate steps as in Art. 5.

This operation of finding derivatives is called **differentiation**. We begin with a function which is itself the sum of two functions.

**7.** Let $u =$ some function of $x$, and $v$ some other function of $x$, and let

$$y=f(x)=u+v.$$

Then, if $x$ takes the increment $\delta x$,

$$y+\delta y=f(x+\delta x)=u+\delta u+v+\delta v,$$

and $$\delta y=\delta u+\delta v;$$

hence $$\frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta x},$$

and in the limit $$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}. \qquad (1)$$

It is evident that the same proof applies to any number of functions connected by plus and minus signs.

A constant quantity, because it is a constant or unvarying quantity, has no increment; and if we attempt to express its derivative, we have nothing to divide by $dx$. This amounts to saying that *the derivative of a constant is zero.*

**8.** Let the given function consist of the product of two functions as expressed by

$$y = f(x) = uv.$$

Then $$y + \delta y = f(x + \delta x) = (u + \delta u)(v + \delta v)$$
$$= uv + v\delta u + u\delta v + \delta u\delta v;$$

hence $$\delta y = v\delta u + u\delta v + \delta u\delta v;$$

therefore $$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \delta u\frac{\delta v}{\delta x},$$

and $$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}. \qquad (2)$$

The last term $\delta u \frac{\delta v}{\delta x}$ is disposed of by observing that as $\delta u$ diminishes without limit, any quantity (except $\infty$) multiplied by $\delta u$ diminishes without limit, and is therefore dropped.

Similarly, if $y = uvw \cdots$ where $u$, $v$, $w$, etc., are functions of $x$,

$$\frac{dy}{dx} = (vw \cdots)\frac{du}{dx} + (uw \cdots)\frac{dv}{dx} + (uv \cdots)\frac{dw}{dx} + \cdots, \text{ etc.}$$

**9.** Taking the last expression of the preceding article, suppose $v = u$, $w = u$, etc., so that $uvw \cdots$ becomes $(u)^n$, $n$ being the number of functions of $x$. Then $y = u^n$, and the expression

$$\frac{dy}{dx} = (vw \cdots)\frac{du}{dx} + (uw \cdots)\frac{dv}{dx} + (uv \cdots)\frac{dw}{dx} + \cdots, \text{ etc.},$$

becomes $$\frac{dy}{dx} = u^{n-1}\frac{du}{dx} + u^{n-1}\frac{du}{dx} + \cdots.$$

This is a polynomial of $n$ terms; hence we can write,

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}. \tag{3}$$

As a special case, if

$$u = x, \quad v = x, \quad w = x, \text{ etc.,}$$

$$y = x^n,$$

and formula (3) becomes

$$\frac{dy}{dx} = nx^{n-1}.$$

**10.** Let $y = f(x) = u^{\frac{p}{q}}$ in which $p$ and $q$ are constant quantities, positive and integral.

Then $$y^q = u^p,$$

and by (3), $$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx};$$

hence $$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{y^{q-1}}\frac{du}{dx}.$$

Eliminating $y$ from this expression by means of the given expression $y = u^{\frac{p}{q}}$, we have

$$\frac{dy}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}. \tag{4}$$

**11.** Let $y = f(x) = u^{-n}$, $n$ being integral and positive. Then $y = \frac{1}{u^n}$, and so $yu^n = 1$. Using the formula for the derivative of the product of two functions, and writing zero for the derivative of unity,

$$u^n \frac{dy}{dx} + ynu^{n-1}\frac{du}{dx} = 0;$$

that is, $$\frac{dy}{dx} = -\frac{ynu^{n-1}}{u^n}\frac{du}{dx} = -\frac{nu^{n-1}}{u^{2n}}\frac{du}{dx},$$

or $$\frac{dy}{dx} = -nu^{-n-1}\frac{du}{dx}. \qquad (5)$$

**12.** Comparing formulas (3), (4), (5), it is seen that if $y = u^n$, in which $u$ is a function of $x$ and $n$ is any constant,

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

The translation of this formula gives, therefore, the only rule that is needed for finding the derivative of a function affected with any constant exponent. It should be noticed, however, that since this expression for the first derivative, $\frac{dy}{dx}$, contains $\frac{du}{dx}$ as a factor, we may require various other rules if we are to find the expression for which $\frac{du}{dx}$ is the symbol. For example, suppose $y = (\log x)^3$. We now know that $\frac{dy}{dx} = 3(\log x)^2$ multiplied by the derivative of $\log x$, whatever it is. What it is we shall learn in a subsequent article. At present we can only write

$$\frac{dy}{dx} = 3(\log x)^2 \frac{d}{dx}\log x.$$

**13.** For the ratio of the indefinitely small increment of the function to the increment of the variable we may of course use $\frac{d}{dx}f(x)$, as well as the symbol $\frac{dy}{dx}$. It should be carefully noticed that the $d$ standing above $dx$ is here, as everywhere, a symbol of *operation* and not of quantity, signifying the differential of $f(x)$. Notice also that an *indicated* operation counts for the same as a performed one. For example, in $\frac{a^2-x^2}{a+x}$ an operation is indicated, and the expression $\frac{a^2-x^2}{a+x}$ has everywhere the same value as $a-x$, the result obtained by actually performing the division. So, for example, $\frac{d}{dx}\pi x^2$ has the same value as $2\pi x$.

**14.** Differentiation is seen to be a tearing down process, whereby we reach an ultimate element of quantity. The reverse operation, one of building up, is known as **integration**; its symbol is $\int$ (long $s$).

As already shown, rules are established for the differentiation of functions, but integration is largely a matter of guesswork and experiment. The test of the correctness of any integration is this: differentiate the result; we should get the given differential form.

Tables of integrals enable the student to write directly the expression corresponding to an indicated integration, so that he need not go through the process of discovering the required expression.

In accordance with what has just been said, we have $\int dy = y$, the symbols $d$, $\int$, neutralizing each other.

Also, if
$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\cdots,$$
$$dy=du+dv+\cdots,$$
and
$$y=u+v+\cdots.$$

If
$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx},$$
$$y=\int(vdu+udv)$$
$$=uv.$$

If
$$\frac{dy}{dx}=u^m\frac{du}{dx},$$
$$y=\frac{u^{m+1}}{m+1}.$$

**15.** The symbol $\int f(x)\,dx$ is known as a general or **indefinite integral.** After discovering a function, say $\phi(x)$, which differentiated will give $f(x)$, we ought to write
$$\int f(x)\,dx=\phi(x)+C,$$
in which $C$ is a quantity primarily undetermined, and known as the **constant of integration.** Since $C$ is a constant,
$$\frac{d}{dx}[\phi(x)+C]=\frac{d}{dx}\phi(x)+\frac{d}{dx}C=f(x),$$
and as a constant term may thus exist in connection with the original function we give the integral the benefit of the doubt and write as stated, $\phi(x)+C$.

**16.** The symbol $\int_a^b f(x)\,dx$ is known as a **definite integral.** Its meaning is this: Find the general integral, which will be some function of $x$, and substitute $b$ for $x$; then substitute $a$ for $x$, and subtract the latter expression from the former. To state the process symbolically, let

$$\int f(x)\,dx = \phi(x);$$

then $$\int_a^b f(x)dx = \phi(x)\Big]_a^b = \phi(b) - \phi(a).$$

$a$ and $b$ are called the **limits** of the integral. The constant $C$ disappears since

$$[\phi(b)+C]-[\phi(a)+C]=\phi(b)-\phi(a).$$

For further discussion of definite integrals, see Art. 73, and for other methods of finding the value of $C$, see Arts. 60, 61, etc.

**Exercises.**

**17.** 1. If $y = nu$, $\frac{dy}{dx} = n\frac{du}{dx}$. Prove this in two ways: (1) by beginning $y + \delta y = n(u + \delta u)$; (2) by regarding $nu$ as a special case of the product of two functions in which one of the functions is a constant (a function by courtesy).

2. If $dy = nf(x)\,dx$, show that

$$y = \int nf(x)\,dx = n\int f(x)\,dx;$$

that is, a constant factor under the integral sign may be removed and written as a coefficient of the integral. Notice that if $dy = nf(x)\,dx$, $\frac{dy}{n} = f(x)\,dx$.

3. If $y = \frac{u}{v}$, show that

$$\frac{dy}{dx} = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}.$$

Translate this formula into a theorem.

4. Prove that if $y = \frac{u}{m}$, $\frac{dy}{dx} = \frac{1}{m}\frac{du}{dx}$, $m$ being a constant. Do this in two ways: (1) by observing that the result in exercise 1 holds when $n$ is a fraction; (2) by making $\frac{u}{m}$ a special case under exercise 3.

5. Prove that if $y = \frac{m}{u}$,

$$\frac{dy}{dx} = -\frac{m\frac{du}{dx}}{u^2}$$

6. Prove that the derivative of the square root of any function is the derivative of the function divided by twice the square root of the function. That is, if $y = \sqrt{u}$,

$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{2\sqrt{u}}.$$

7. Show that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$; that is, the limits may be reversed if the sign of the integral is changed.

8. If $y$ is the surface of a sphere whose radius is $x$,

$$\frac{dy}{dx} = 8\,\pi x.$$

9. Show that the ratio of the differential of the circumference of a circle is to the differential of its radius as $2\,\pi : 1$.

**18.** Thus far $y$ has been in each instance an explicit function of $x$. Suppose now that $x$ and $y$ are so combined in any expression that $y$ is only implicitly a function of $x$; for example, as in the expression $ax + by + c = 0$. This statement is the equivalent of the explicit statement $y = \frac{-ax-c}{b}$. A little algebraic consideration will show that it is unnecessary to solve for $y$ before proceeding to find $\frac{dy}{dx}$. We may differentiate immediately, and then solve for $\frac{dy}{dx}$. Thus, if $ax + by + c = 0$, differentiating term by term with $x$ as the fundamental variable, we have $a + b\frac{dy}{dx} = 0$; and hence $\frac{dy}{dx} = -\frac{a}{b}$, which is the same result that we get by differentiating $y = \frac{-ax-c}{b}$.

### Exercises.

**19.** 1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; show that $\frac{dy}{dx} = \pm\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}$.

Differentiating immediately,

$$\frac{2x}{a^2} + \frac{2y\frac{dy}{dx}}{b^2} = 0;$$

solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}.$$

$y$ may now be eliminated by using its value

$$\pm\frac{b}{a}\sqrt{a^2-x^2},$$

obtained from the given equation.

2. $x^3 + y^3 - 3axy = 0$; find $\frac{dy}{dx}$.

3. $x^3 + y^3 - 3axy + a^3 = 0$; find $\frac{dy}{dx}$.

4. $pv = c$; show that $\frac{dp}{dv} = -\frac{c}{v^2}$ if $p = f(v)$.

By formula (2), $$v\frac{dp}{dv} + p = 0;$$

hence, $$\frac{dp}{dv} = -\frac{p}{v} = -\frac{c}{v^2}.$$

5. $pv^k = c'$; show that $\frac{dp}{dv} = -\frac{kc'}{v^{k+1}}$.

Here, and also in exercise 4, it is just as well to solve for $p$ before beginning to differentiate. We have $p = \frac{c'}{v^k}$, so that

$$\frac{dp}{dv} = -\frac{kc'v^{k-1}}{v^{2k}} = -\frac{kc'}{v^{k+1}}.$$

6. $F = c\frac{Mm}{r^2}$, in which $c$, $M$, $m$ are constants; show that

$$\frac{dF}{dr} = -\frac{2\,cMm}{r^3}.$$

It is often desirable to use other letters besides $x$ and $y$ to denote the variable quantities. The student should therefore accustom himself at the outset to such symbols as those given in exercises 4, 5, 6.

**20.** If $f(x)$ is a varying angle, it is clear that any trigonometric function of the angle must also vary. We have now to find $\frac{dy}{dx}$, when $y$ represents each one of the trigonometric functions in turn.

Suppose $y = \sin f(x)$. Let $f(x) = u$; then $y = \sin u$,

and $$\frac{\delta y}{\delta x} = \frac{\sin(u + \delta u) - \sin u}{\delta x}$$

Put $$u + \delta u = \alpha, \text{ and } u = \beta;$$

then, since $\sin\alpha - \sin\beta = 2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)$,

$$\begin{aligned}\delta y &= \sin(u+\delta u) - \sin u = \sin\alpha - \sin\beta \\ &= 2\sin\tfrac{1}{2}\delta u\cos\tfrac{1}{2}(2u+\delta u) \\ &= 2\cos(u+\tfrac{1}{2}\delta u)\sin\tfrac{1}{2}\delta u;\end{aligned}$$

hence $$\frac{\delta y}{\delta x} = \cos(u+\tfrac{1}{2}\delta u)\frac{\sin\frac{1}{2}\delta u}{\frac{1}{2}\delta u}\cdot\frac{\delta u}{\delta x}.$$

But when an angle diminishes without limit, we may write the angle itself for its sine; so we now have

$$\frac{dy}{dx} = \frac{d}{dx}\sin u = \cos u\frac{du}{dx}. \tag{6}$$

Let $$y = \cos u = \sin\left(\frac{\pi}{2} - u\right);$$

then $$\frac{dy}{dx} = \frac{d}{dx}\sin\left(\frac{\pi}{2}-u\right) = \cos\left(\frac{\pi}{2}-u\right)\frac{d}{dx}\left(\frac{\pi}{2}-u\right);$$

therefore $$\frac{dy}{dx} = \frac{d}{dx}\cos u = -\sin u\frac{du}{dx}. \tag{7}$$

Let $$y = \tan u = \frac{\sin u}{\cos u}.$$

Using exercise 3, Art. 17, together with formulas (6) and (7),

$$\frac{dy}{dx} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx} = \sec^2 u\frac{du}{dx};$$

therefore $$\frac{dy}{dx} = \frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}. \tag{8}$$

Let $$y = \cot u = \frac{\cos u}{\sin u}.$$

$$\frac{dy}{dx} = \frac{-\sin^2 u - \cos^2 u}{\sin^2 u}\frac{du}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx};$$

therefore $$\frac{dy}{dx} = \frac{d}{dx}\cot u = -\operatorname{cosec}^2 u \frac{du}{dx}. \qquad (9)$$

Let $$y = \sec u = \frac{1}{\cos u};$$

then, using exercise 5, Art. 17,

$$\frac{dy}{dx} = \frac{\sin u \dfrac{du}{dx}}{\cos^2 u} = \tan u \sec u \frac{du}{dx},$$

and we have

$$\frac{dy}{dx} = \frac{d}{dx}\sec u = \tan u \sec u \frac{du}{dx}. \qquad (10)$$

Let $$y = \operatorname{cosec} u = \frac{1}{\sin u};$$

$$\frac{dy}{dx} = \frac{-\cos u \dfrac{du}{dx}}{\sin^2 u} = -\cot u \operatorname{cosec} u \frac{du}{dx},$$

and therefore

$$\frac{dy}{dx} = \frac{d}{dx}\operatorname{cosec} u = -\cot u \operatorname{cosec} u \frac{du}{dx}. \qquad (11)$$

**21.** In formulas (6) to (11) inclusive we have the ratio of the differential of the trigonometric function to the differential of the angle $x$. These formulas, which are remarkable for their simplicity, should be translated and committed to memory. We may next regard the trigonometric function as varying by equal increments, and thereby causing a change in the angle. From this point of view we have to find the ratio of the differential of the angle to the differential of the trigonometric function.

**22.** Let $y = \sin^{-1} u$; then $u = \sin y$.

Hence, by formula (6),

$$\frac{du}{dx} = \cos y \frac{dy}{dx};$$

that is,
$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\cos y}.$$

But
$$\cos y = \sqrt{1 - u^2};$$

therefore
$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1} u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}. \tag{12}$$

Let $y = \cos^{-1} u$; then $u = \cos y$, and proceeding as before, we find

$$\frac{dy}{dx} = \frac{d}{dx}\cos^{-1} u = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}. \tag{13}$$

Let
$$y = \tan^{-1} u; \text{ then } u = \tan y,$$

and
$$\frac{du}{dx} = \sec^2 y \frac{dy}{dx}; \text{ that is, } \frac{dy}{dx} = \frac{\frac{du}{dx}}{\sec^2 y};$$

therefore
$$\frac{dy}{dx} = \frac{d}{dx}\tan^{-1} u = \frac{\frac{du}{dx}}{1+u^2}; \tag{14}$$

similarly, if $y = \cot^{-1} u$,

$$\frac{dy}{dx} = \frac{d}{dx}\cot^{-1} u = -\frac{\frac{du}{dx}}{1+u^2}. \tag{15}$$

Let $y = \sec^{-1} u. \quad u = \sec y,$

and $\dfrac{du}{dx} = \tan y \ \sec y \dfrac{dy}{dx};$

that is, $$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\tan y \ \sec y}$$

But since $u = \sec y$, $\tan y = \sqrt{u^2 - 1}$; and we have

$$\frac{dy}{dx} = \frac{d}{dx} \sec^{-1} u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}. \qquad (16)$$

Finally, if $y = \operatorname{cosec}^{-1} u$, we find that

$$\frac{dy}{dx} = \frac{d}{dx} \operatorname{cosec}^{-1} u = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}. \qquad (17)$$

All these operations of differentiating may be reversed, so that if

$$\frac{dy}{dx} = \cos u \frac{du}{dx}, \ y = \int \cos u \, du = \sin u, \text{ etc.}$$

**23.** It remains to find $\dfrac{dy}{dx}$ when $y = e^x$, $a^x$, $\log x$, $\log u$, $e^u$, $a^u$, $u$ being, as before, equal to $f(x)$.

From algebra* we have

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots + \frac{x^n}{\lfloor n} + \cdots,$$

in which $e$ is the number 2.7182818284 . . . forming the base of the Napierian system of logarithms. If $y = e^x$,

$$\frac{dy}{dx} = \frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\right).$$

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*See Hall and Knight's *Elementary Algebra*, Art. 537. Edition of 1896.

Performing the operation indicated, that is, differentiating this series term by term,

$$\frac{d}{dx}e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots ;$$

but this result is the original series which $e^x$ equals; therefore

$$\frac{dy}{dx} = \frac{d}{dx}e^x = e^x. \qquad (18)$$

This result is unique, being the only case known in which the derivative of a function is the function itself.

We have also from algebra

$$a^x = 1 + x \log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \cdots ;$$

therefore

$$\frac{d}{dx}a^x = \log_e a\left[1 + x \log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \cdots\right]$$
$$= a^x \log_e a.$$

Hence, if $y = a^x$,

$$\frac{dy}{dx} = \frac{d}{dx}a^x = a^x \log_e a. \qquad (19)$$

**24.** Let $y = \log_e x$; then $x = e^y$; and if we regard $x$ as a function of $y$, we have by formula (18),

$$\frac{dx}{dy} = e^y ; \text{ that is, } \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x};$$

therefore $$\frac{dy}{dx} = \frac{d}{dx}\log_e x = \frac{1}{x}. \qquad (20)$$

To find $\frac{dy}{dx}$ when $y = \log_e f(x) = \log_e u$, we notice that $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$ and in the limit

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Now if $y = \log_e u$, $\frac{dy}{du} = \frac{1}{u}$, by formula (20),

and therefore $$\frac{dy}{dx} = \frac{d}{dx}\log_e u = \frac{1}{u}\frac{du}{dx}. \qquad (21)$$

If $y = e^u$, $\log_e y = u$

and $$\frac{du}{dx} = \frac{d}{dx}\log y = \frac{1}{y}\frac{dy}{dx},$$ by formula (21),

hence $$\frac{dy}{dx} = y\frac{du}{dx} = e^u\frac{du}{dx};$$

so that we have $$\frac{dy}{dx} = \frac{d}{dx}e^u = e^u\frac{du}{dx}. \qquad (22)$$

Finally, if $y = a^u$, $\log y = u \log a$, and

$$\frac{du}{dx} = \frac{1}{\log a}\frac{d}{dx}\log y = \frac{1}{y \log a}\frac{dy}{dx};$$

hence $$\frac{dy}{dx} = y\frac{du}{dx}\log a = a^u\frac{du}{dx}\log a,$$

and therefore $$\frac{dy}{dx} = \frac{d}{dx}a^u = a^u\frac{du}{dx}\log a. \qquad (23)$$

Formulas (1) to (23), together with the corresponding integration formulas, are collected in Chapter V.

**25.** Since first derivatives are themselves generally functions of the fundamental variable, their first derivatives may in turn be found. These last are called **second derivatives** of the original functions. Second derivatives are indicated by the symbols $f''(x)$, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d^2y}{dx^2}$. It is to be carefully noticed that "$d^2$," like $d$, is not a symbol of quantity, but of operation. The $d^2$ should never be read "$d$ square," but "second $d$." Since $\frac{d^2y}{dx^2}$ always means $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, it is best to use the latter form until there is no danger of misunderstanding the symbol $\frac{d^2y}{dx^2}$. The derivative of the second derivative is called the third derivative and is written $\frac{d^3y}{dx^3}$; the next is written $\frac{d^4y}{dx^4}$, and so on.

**26.** In what has preceded we have assumed one fundamental variable; but reference to common examples shows us that we may have a function of two or more independent variables. For instance, crops vary not only with the amount of fertilizer used, but also with the amount of sunshine and moisture.

If $z$ is a function of two independent variables $x$ and $y$, expressed by writing $u = f(x, y)$, we may differentiate, supposing $x$ to vary and $y$ to remain constant, or we may suppose $y$ to vary and $x$ to remain constant. In the former case we have $\frac{dz}{dx}$; and in the latter, $\frac{dz}{dy}$. These ratios are known as **partial differential coefficients**, and to indicate this we may use the parenthesis, writing

$\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$. When we suppose $x$ and $y$ to vary simultaneously the corresponding change in the function is called the **total differential.** As an example of partial differentials suppose we have $pv = nt$, where $p$ and $v$ are pressure and volume as before and $t$ is the absolute temperature of a gas. Suppose $t$ varies while $v$ remains constant; then $\left(\frac{dp}{dt}\right) = \frac{n}{v}$. Again, let $v$ vary while $t$ remains constant, $\left(\frac{dp}{dv}\right) = \frac{-nt}{v^2}$, as in exercise 4, Art. 19.

**27.** Having $z = f(x, y)$, let us differentiate $z$ with respect to $x$ and then differentiate the result with respect to $y$. The order of the steps is indicated by

$$\frac{d}{dy}\left(\frac{dz}{dx}\right) \text{ or } \frac{d^2z}{dydx}.$$

The reverse order is indicated by

$$\frac{d}{dx}\left(\frac{dz}{dy}\right) \text{ or } \frac{d^2z}{dxdy}.$$

We proceed to show that

$$\frac{d}{dy}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dz}{dy}\right);$$

that is, we get the same result in whichever order we proceed.

Let $x$ take the increment $\delta x$ while $y$ remains constant;

then $$\frac{\delta z}{\delta x} = \frac{f(x + \delta x, y) - f(x, y)}{\delta x}.$$

Now let $y$ in this result take the increment $\delta y$ while $x$ remains constant.

$$\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta x}\right)$$
$$=\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)-f(x+\delta x, y)+f(x,y)}{\delta y\delta x}.$$

Reversing the order, we have

$$\frac{\delta z}{\delta y}=\frac{f(x,\ y+\delta y)-f(x,y)}{\delta y},$$

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right)$$
$$=\frac{f(x+\delta x, y+\delta y)-f(x+\delta x, y)-f(x, y+\delta y)+f(x,y)}{\delta x\delta y}.$$

Hence $$\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta x}\right)=\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right);$$

and in the limit

$$\frac{d}{dy}\left(\frac{dz}{dx}\right)=\frac{d}{dx}\left(\frac{dz}{dy}\right);\quad \text{or}\quad \frac{d^2z}{dydx}=\frac{d^2z}{dxdy}.$$

In any scientific investigation in which the calculus is used the context must show what and how many variables are involved, and what partial differential coefficients will occur. For example, Carnot's Principle, with its applications as presented in thermodynamics, affords an abundance of cases of these partial differential coefficients.

**28.** Ordinary text-books in algebra and trigonometry usually give methods for expanding $(a+x)^m$, $e^x$, $a^x$, $\log(1+x)$, $\sin x$, etc., into series in ascending powers of $x$. We can now establish one general theorem by means of which these functions and all similar ones may be expanded.

We first notice that if $y=f(z+x)$ and we differentiate regarding $x$ as a variable and $z$ as a constant, the result is just the same as if we should differentiate with $z$ for the variable and $x$ as a constant. That is, if $y=f(z+x)$, $\left(\frac{dy}{dx}\right)=\left(\frac{dy}{dz}\right)$. This is obviously true if we consider that it makes no difference whether we change the function by changing $z$ or $x$.

Suppose

$$f(z+x)=A+Bx+Cx^2+Dx^3+Ex^4+\cdots, \qquad (a)$$

in which $A$, $B$, $C$, etc., are functions of $z$ and not of $x$. Let us now differentiate successively the first member of equation $(a)$ with respect to $z$, and the second member with respect to $x$, and put $x=0$ after each differentiation.

Then, since $\frac{d}{dz}f(z+x)=\frac{d}{dx}f(z+x)$,

$$\frac{d}{dz}f(z+x)=f'(z+x)$$
$$=B+2Cx+3Dx^2+4Ex^3+\cdots,$$

and $$f'(z)=B.$$

$$f''(z+x)=2C+2\cdot 3Dx+3\cdot 4Ex^2+\cdots,$$

and $$f''(z)=2C.$$

$$f'''(z+x)=2\cdot 3D+2\cdot 3\cdot 4Ex+\cdots,$$

and $$f'''(z)=2\cdot 3D.$$

. . . . . . .

Also, if $x=0$ in equation $(a)$, $f(z)=A$.

We now have

$$A=f(z),\ B=f'(z),\ C=\tfrac{1}{2}f''(z),\ \text{etc.}$$

Putting these values into the assumed series $(a)$,

$$f(z+x)=f(z)+f'(z)x+\frac{f''(z)x^2}{\lfloor 2}+\frac{f'''(z)x^3}{\lfloor 3}+\cdots+\frac{f^n(z)x^n}{\lfloor n}+\cdots.$$

This formula is **Taylor's theorem.** It enables us to expand functions of the sum of two variables in ascending powers of one of the variables, combined with finite coefficients depending on the other variable.

**29.** Suppose we have a function of one variable and wish to expand it into a series. Following the method of the preceding article, assume

$$f(x)=A+Bx+Cx^2+Dx^3+Ex^4\cdots. \qquad (b)$$

Differentiating successively and putting $x=0$ after each differentiation, we have

$$f'(x)=B+2Cx+3Dx^2+4Ex^3+\cdots,$$

and $\qquad f'(x)]_0=B.$

$$f''(x)=2C+2\cdot 3Dx+3\cdot 4Ex^2+\cdots,$$

and $\qquad f''(x)]_0=2C.$

$$f'''(x)=2\cdot 3D+2\cdot 3\cdot 4Ex+\cdots,$$

$$f'''(x)]_0=2\cdot 3D.$$

. . . . . . .

Also, $$f(x)]_0 = A.$$

The assumed coefficients $A$, $B$, etc., are thus determined, for we have

$$A = f(x)]_0, \text{ or, as it is usually written, } f(0);$$

$$B = f'(x)]_0 = f'(0);$$

$$C = \frac{f''(x)]_0}{2} = \frac{f''(0)}{2};$$

$$D = \frac{f'''(x)]_0}{\lfloor 3} = \frac{f'''(0)}{\lfloor 3}, \text{ etc.}$$

Putting these values into the assumed series ($b$),

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{\lfloor 2} + \frac{f'''(0)x^3}{\lfloor 3} + \cdots + \frac{f^n(0)x^n}{\lfloor n} + \cdots.$$

This formula is **Maclaurin's theorem.**

It will be observed that if $z$ is made equal to zero in Taylor's theorem, we have Maclaurin's theorem. The latter may therefore be regarded as a special case under Taylor's theorem.

**30.** Suppose $$f(x) = (a + x)^m;$$

let us expand this function according to Maclaurin's theorem.

If $x = 0$, the function becomes $a^m$.

Further, $f'(x) = m(a+x)^{m-1}$,

and $f'(0) = m(a+x)^{m-1}\big]_0 = ma^{m-1}$.

$$f''(x) = m(m-1)(a+x)^{m-2},$$

$$f''(0) = m(m-1)(a+x)^{m-2}\big]_0 = m(m-1)a^{m-2}.$$

$$f'''(x) = m(m-1)(a+x)^{m-3},$$

and hence

$$f'''(0) = m(m-1)(m-2)(a+x)^{m-3}\big]_0$$
$$= m(m-1)(m-2)a^{m-3}.$$

. . . . . . . .

Therefore we have

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)a^{m-2}x^2}{\underline{|2}}$$
$$+ \frac{m(m-1)(m-2)a^{m-3}x^3}{\underline{|3}} + \cdots$$
$$+ \frac{m(m-1)\cdots(m-n+2)a^{m-(n-1)}x^{n-1}}{\underline{|n-1}} + \cdots.$$

This formula will be recognized as the **binomial theorem.** It provides for the expansion of a binomial affected with any constant exponent.

**31.** A series must be known to be a *converging* series before any practical use can be made of it. The simplest tests for convergency are given in algebra text-books.* If a series is found to be diverging, it is

* See Hall and Knight's *Elementary Algebra*, Arts. 470–477.

rejected for such values of the variable as make it diverging; or it is transformed into a series which converges and, if possible, into one which converges rapidly, in order that only a few terms need be used.

**32.** If one function is divided by another, as $\frac{f(x)}{\phi(x)}$, it sometimes happens that the functions are of such a nature that upon evaluating them for some particular quantity each function reduces to zero, so that we have $\frac{0}{0}\cdot$ The question arises: what does this expression mean, and what is its value?

Students often say that $\frac{0}{0}$ must be unity; sometimes they are inclined to think it is zero. In some instances the first view is correct; in others, the second; in others still, a value will be found which is neither unity nor zero. Now it is evident that if we can find a limit which the ratio $\frac{f(x)}{\phi(x)}$ is approaching as $x$ approaches nearer and nearer to that value which makes $f(x)$ and $\phi(x)$ each equal to zero, we have caught the correct value of $\frac{0}{0}\cdot$ We proceed to find a general expression for this limit.

Suppose $x$ to take the increment $\delta x$; then by Taylor's theorem,

$$\frac{f(x+\delta x)}{\phi(x+\delta x)}=\frac{f(x)+f'(x)\delta x+\frac{f''(x)}{2}(\delta x)^2+\cdots}{\phi(x)+\phi'(x)\delta x+\frac{\phi''(x)}{2}(\delta x)^2+\cdots}\cdot \qquad (1)$$

Let $a$ be the quantity that makes both $f(x)$ and $\phi(x)$ equal to zero. Substituting $a$ for $x$,

$$\frac{f(a+\delta x)}{\phi(a+\delta x)}=\frac{0+f'(a)\delta x+\frac{f''(a)}{2}(\delta x)^2+\cdots}{0+\phi'(a)\delta x+\frac{\phi''(a)}{2}(\delta x)^2+\cdots} \qquad (2)$$

Dividing both numerator and denominator of the second member of this expression by $\delta x$,

$$\frac{f(a+\delta x)}{\phi(a+\delta x)}=\frac{f'(a)+\frac{f''(a)}{2}\delta x+\cdots}{\phi'(a)+\frac{\phi''(a)}{2}\delta x+\cdots}.$$

Finally, when $\delta x$ becomes $dx$,

$$\frac{f(a)}{\phi(a)}=\frac{f'(a)}{\phi'(a)}.$$

Hence, if $\frac{f(x)}{\phi(x)}$ becomes $\frac{0}{0}$ when evaluated for any quantity as $a$, the value of this indeterminate form is $\frac{f'(x)}{\phi'(x)}$ evaluated for $a$.

If it should happen that $\left.\frac{f'(x)}{\phi'(x)}\right]_a=\frac{0}{0}$, we divide both numerator and denominator of equation (2) by $\delta x$ again and have

$$\frac{f(a)}{\phi(a)}=\frac{0}{0}=\frac{f''(a)}{\phi''(a)}.$$

If $\frac{f(x)}{\phi(x)}$ becomes $\frac{\infty}{\infty}$ when evaluated for any quantity as $a$, this expression may be determined as in the first case by observing that if

$$\frac{f(x)}{\phi(x)}=\frac{\infty}{\infty},\quad \frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}}=\frac{0}{0}.$$

If $f(x)\phi(x)=\infty\cdot 0$ when evaluated for some quantity, $\frac{\phi(x)}{\frac{1}{f(x)}}=\frac{0}{0}$, which may be treated as above.

If $f(x)-\phi(x)=\infty-\infty$ for some quantity, it should be transformed into a fraction which takes the form $\frac{0}{0}$ and then determined.

**33.** If $y=[f(x)]^{\phi(x)}$, $\log y=\phi(x)\log f(x)$; and this is indeterminate whenever one of the factors becomes zero and the other infinite for the same value of $x$.

(1) Suppose $\phi(x)=0$, and $\log f(x)=\pm\infty$; then $f(x)=\infty$ or 0. Consequently $[f(x)]^{\phi(x)}$ becomes indeterminate when for some value of $x$ it takes the form $0^{\circ}$ or $\infty^{\circ}$.

(2) Suppose $\phi(x)=\pm\infty$, and $\log f(x)=0$; then $f(x)=1$, and $[f(x)]^{\phi(x)}$ gives the indeterminate forms $1^{\infty}$ and $1^{-\infty}$.

Hence, if we have any of the indeterminate forms $0^{\circ}$, $\infty^{\circ}$, $1^{\pm\infty}$, as the result of evaluating $[f(x)]^{\phi(x)}$ for some quantity, we change the exponential function to the corresponding logarithmic function, and then reduce to the form $\frac{0}{0}$, which is dealt with under the first case.

## Exercises.

**34.** 1. Show that $\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u\,\frac{du}{dx}$ through the relation $\cot u = \tan\left(\frac{\pi}{2} - u\right)$.

2. If $y = [f(x)]^{\phi(x)} = u^v$, show that

$$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + (\log u)\,u^v\frac{dv}{dx}.$$

Take the logarithmic form, $\log y = v \log u$, and differentiate.

3. $y = x^x;\quad \frac{dy}{dx} = x^x(1 + \log x).$

4. $y = e^{-\frac{1}{x}};\quad \frac{dy}{dx} = \frac{1}{x^2 e^{\frac{1}{x}}}.$

5. $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right);\quad \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$

6. $N = e\lambda \tan F - \log\tan(45° + \frac{1}{2}F)$, in which the variables are $N$ and $F$. Show that

$$\frac{dN}{dF} = \frac{\lambda(e - \cos F)}{\cos^2 F}.$$

— Watson's *Theoretical Astronomy*, p. 69.

7. Verify the following expansions by means of Maclaurin's theorem:

(i) $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots + \frac{x^n}{\underline{|n}} + \cdots.$

(ii) $a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \cdots.$

(iii) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots(-1)^{n-1}\frac{x^n}{n}\cdots.$

(iv) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots.$

(v) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots.$

Show that (v) might be derived directly from (iv), since

$$\frac{d}{dx} \sin x = \cos x.$$

**8.** Show that $$\left.\frac{\sin x}{x}\right]_0 = 1.$$

In this case $f'(x) = \cos x$ and $\phi'(x) = 1$; $\left.\frac{\cos x}{1}\right]_0 = 1.$

**9.** Show that $$\left.\frac{a^x - b^x}{x}\right]_0 = \log \frac{a}{b}.$$

**10.** Show that $$\left.\frac{x \log (1+x)}{1 - \cos x}\right]_0 = 2.$$

We have $$\frac{f'(x)}{\phi'(x)} = \frac{\log(1+x) + x\left(\frac{1}{1+x}\right)}{\sin x};$$

but this expression evaluated for 0 is $\frac{0}{0}$ as before. Hence we proceed to the second derivatives and have

$$\frac{f''(x)}{\phi''(x)} = \frac{\frac{1}{1+x} + \frac{1}{1+x} + x\left(\frac{-1}{(1+x)^2}\right)}{\cos x},$$

and this equals 2 when evaluated for 0.

**11.** Show that $\frac{1 - \sin x - 2\sin^2 x}{1 - 3\sin x + 2\sin^2 x} = 3$ when $x = 30°$.

**12.** Expand $\sin^{-1} x$. Using Maclaurin's theorem,

$$f(x) = \sin^{-1} x = f(0) + f'(0)\, x + \frac{f''(0)}{\underline{|2}}\, x^2 + \frac{f'''(0)}{\underline{|3}}\, x^3 + \cdots$$

$$f(0) = 0;\ f'(x) = \frac{1}{\sqrt{1-x^2}};\ f'(0) = 1.$$

$$f''(x) = \frac{x}{(1-x^2)^{\frac{3}{2}}};\ f''(0) = 0.$$

$$\left. f'''(x) \right]_0 = \left. \frac{(1-x^2)^{\frac{1}{2}} - \frac{3}{2}\, x}{(1-x^2)^2} \right]_0 = 1.$$

Therefore $$\sin^{-1} x = x + \frac{x^3}{\underline{|3}} + \cdots$$

**13.** Show that $2\, r \sin^{-1} \dfrac{a}{2\, r} = a\left(1 + \dfrac{a^2}{24\, r^2} + \cdots\right)$.

— Thomson and Tait's *Nat. Phil.*, Vol. 1, Art. 131.

Substituting $\dfrac{a}{2\, r}$ for $x$ in the expansion of $\sin^{-1} x$, and multiplying by $2\, r$, we have the result

$$a\left(1 + \frac{a^2}{24\, r^2} + \cdots\right).$$

**14.** Expand $\sin^2 \dfrac{\theta}{2}$ to the second power term inclusive.

*Ans.* $\dfrac{\theta^2}{4}$.

**15.** Given $\int x \log x \, dx$; perform the operation indicated.

We have $$d(uv) = udv + vdu; \qquad \text{(formula 2)}$$

hence, $$uv = \int udv + \int vdu, \qquad \text{(Art. 14)}$$

or, $$\int udv = uv - \int vdu.$$

In the present case let

$$u = \log x; \text{ then } dv = x\,dx, \text{ and } du = \frac{1}{x}.$$

Integrating $dv$, $\quad v = \frac{x^2}{2}$, and $uv = \frac{x^2}{2}\log x$;

therefore $\displaystyle\int x \log x\,dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2}\frac{dx}{x}$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}.$$

This process is known as **integration by parts.**

**16.** Use the method of example 15 in the following examples:

(i) $\displaystyle\int x \cos x\,dx = x \sin x + \cos x.$

(ii) $\displaystyle\int e^{ax} x\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$

(iii) $\displaystyle\int \sin^{-1} x\,dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

(iv) $\displaystyle\int \log x\,dx = x \log x - x.$

**17.** Given the following integrals, to find their values:

(i) $\displaystyle\int (a - x)^n\,dx.$

(ii) $\displaystyle\int \tan x\,dx.$

Let $\cos x = u$; then $du = -\sin x\,dx$, and

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\log u = -\log \cos x.$$

(iii) $\int \sin x \cos x \, dx.$

This may be written $\int \frac{1}{2} \sin 2x \, dx$, which equals $-\frac{1}{4}\cos 2x$.

(iv) $\int x\sqrt{a^2 - x^2}\, dx.$

Let $u = \sqrt{a^2 - x^2}$; then $\int x\sqrt{a^2 - x^2}\, dx = -\int u^2 du$

$= -\frac{u^3}{3} = -\frac{(a^2 - x^2)^{\frac{3}{2}}}{3}.$

(v) $\int \sqrt{a^2 - x^2}\, dx.$

Let $x = a \sin u$; then $\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2 u du$

$= \frac{a^2}{2}\int (1 + \cos 2u)\, du = \frac{a^2}{2}(u + \frac{1}{2}\sin 2u)$

$= \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2 - x^2}\right) = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}.$

(vi) $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx.$

This may be written

$$\int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}} dx, \text{ which equals}$$

$$\int \frac{a^2}{\sqrt{a^2 - x^2}} dx - \int \sqrt{a^2 - x^2}\, dx.$$

Therefore

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = a^2 \sin^{-1}\frac{x}{a} - \left(\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}\right).$$

(vii) $\int \frac{dx}{\sqrt{a^2 + x^2}}$.

Let
$$u - x = \sqrt{a^2 + x^2};$$

then
$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{du}{u} = \log u = \log(x + \sqrt{a^2 + x^2}).$$

(viii) $\int \sqrt{a^2 + x^2}\, dx$.

Integrate by parts, letting $u = \sqrt{a^2 + x^2}$;

then
$$dv = dx,\ v = x,\ du = \frac{x\,dx}{\sqrt{a^2 + x^2}},\ uv = x\sqrt{a^2 + x^2},$$

and the formula
$$\int u dv = uv - \int v du$$

becomes

$$\int \sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$= x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\, dx + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx.$$

Transposing the middle term and dividing by 2,

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2 + x^2}}$$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log(x + \sqrt{a^2 + x^2}).$$

**35.** It should be remembered that the system of logarithms used in this chapter is the Napierian. Whenever differentiation or integration gives rise to an expression in which a logarithmic factor occurs, the equation containing this factor must be multiplied through by 0.43429448 ⋯, the modulus of the common system, before it can be used in computations involving common logarithms. For example, in developing the theory for determining the place of a comet moving in a hyperbolic orbit we encounter the equation

$$k\sqrt{p}\,dt = a^2 \tan\psi\left[\frac{1}{2}e\left(1+\frac{1}{\sigma^2}\right)-\frac{1}{\sigma}\right]d\sigma$$

in which $t$ and $\sigma$ are the variables. This is to be integrated between the limits $T$ and $t$; so we have

$$k\sqrt{p}\int_t^T dt = \int a^2 \tan\psi\left[\frac{1}{2}e\left(1+\frac{1}{\sigma^2}\right)-\frac{1}{\sigma}\right]d\sigma$$

$$= a^2 \tan\psi\left[\int\frac{1}{2}e\left(1+\frac{1}{\sigma^2}\right)d\sigma - \int\frac{1}{\sigma}\,d\sigma\right];$$

hence $$k\sqrt{p}\,(T-t) = a^2 \tan\psi\left[\frac{1}{2}e\left(\sigma-\frac{1}{\sigma}\right)-\log_e\sigma\right].$$

Common logarithms cannot be used in this equation or in any modified form of it without first introducing the modulus of the common system as a factor throughout.

# CHAPTER II.

## THE GRAPH.

**36.** The changes in a function corresponding to changes in its variable may be graphically shown in the following way:

Draw a horizontal line with another line at right angles to it. Call the horizontal line $XX'$ or the **x-axis**; the vertical line $YY'$ or the **y-axis**; and their point of intersection $O$ or the **origin**.

If $y$ is some specified function of $x$, give a number of convenient values to $x$ and find the accompanying values for $y$. Beginning at $O$ as the zero point, lay off with any convenient unit of length the positive values of $x$ to the right on the $x$-axis, and the negative values to the left on this axis. At the end, remote from $O$, of this line, which represents a value of $x$, draw a perpendicular (using the same unit of length) to represent the corresponding value of $y$. The perpendicular is to be drawn upward from the $x$-axis in case $y$ is positive, and downward when $y$ is negative.

In this way locate a point for each pair of values of $x$ and $y$. If many values be given to $x$, — any two consecutive values differing but little from each other, — we shall have a correspondingly large number of points with small distances separating them. Connecting all the points in order, we have a continuous line, straight or curved, called a **graph** or **locus**. The values of $x$ are called **abscissas**, and the values of $y$ **ordinates**. The two together are known as **coördinates**.

To illustrate, suppose $y = x + 2$. When $x=0$, $y=2$; when $x = 1$, $y = 3$; when $x = -2$, $y = 0$; etc. When $x = 0$, we have no distance to measure off on the $x$-axis, and since $y = +2$ we measure upward two units, thus locating the point $P_1$. Measuring one unit to the right and three upward, we have the point $P_2$. Locating a number of points in this way and then connecting them, the result looks like a straight line. At any rate we have not been able to get any apparent bends or corners — provided the plotting has been accurately done. This line presents to the eye the way $y$ changes as $x$ changes when $y = x + 2$. The vertical lines representing the values of $y$ seem to get steadily longer as $x$ increases.

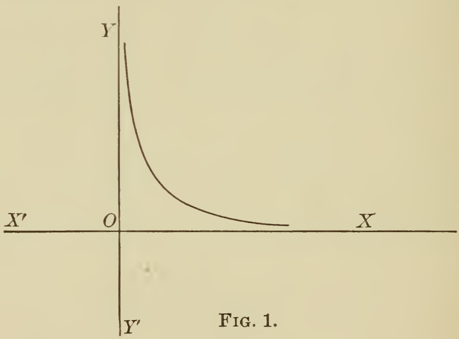


Fig. 1.

As another illustration, let us take the isotherm equation $y = \frac{c}{x}$ (see Art. 5). Suppose $c$ is unity, so that $y = \frac{1}{x}\cdot$ When $x = 1$, $y = 1$; when $x = 2$, $y = \frac{1}{2}$; when $x = \frac{1}{2}$, $y = 2$; etc. Locating these points and drawing a smooth curve through them, the graph appears as in Fig. 1. Two things are clear in regard to this graph: (1) it is related to the $y$-axis precisely as it is to the $x$-axis; (2) as $x$ increases without limit, $y$ diminishes without limit, so that the points are nearer and nearer to the $x$-axis. The graph therefore shows what the equation says; namely, that as the volume becomes indefinitely great the pressure becomes indefinitely small; and conversely, if the volume could be diminished without limit, the pressure would be indefinitely great.

We further observe that when $x$ is negative, $y$ is negative; and thus the complete graph includes a branch in the diagonally opposite corner $X'OY'$ (Art. 123). But this second branch represents no actual pressures and volumes, because pressures and volumes are positive. We shall find numerous instances of equations in which the variables, abstractly viewed, have a wider range of values than the values possible for the concrete quantities under consideration.

**37.** If we like, we may think of a graph as the *path of a point* which moves from one determined point to the next one, and thence to the next one in order. The equation $y = x + 2$, for example, merely says that the point moves so that its ordinate is all the time equal to its abscissa increased by the constant 2. When the graph is thus looked upon as the path of a moving point, the variable coördinates $x$ and $y$ are called **current coördinates.** Any equation in two variables may be said to express the *law* of the point's motion in the plane.

For brevity we shall speak of "the curve $y=f(x)$," instead of saying "the curve which the equation $y=f(x)$ represents."

**38.** Suppose the moving point describes the arc $CC'$ of the graph or curve $y = f(x)$. Let $P$ be any point in the path and $Q$ another point. As the moving point goes from $P$ to $Q$, its abscissa changes from $x$ to $x + \delta x$, and its ordinate from $y$ to $y + \delta y$. Draw $PL$ parallel to the $x$-axis. Then $PL = \delta x$ and $LQ = \delta y$. Let $TT'$ be the chord (produced) passing through the points $P$, $Q$. $\frac{LQ}{PL}$ is the tangent of the angle which the line $TT'$ makes with the $x$-axis. Now suppose $\delta x$ and $\delta y$ to become indefinitely small. $P$ and $Q$ must approach indefinitely near to each other, the chord becomes a tangent, and $\frac{LQ}{PL} = \frac{dy}{dx}$.

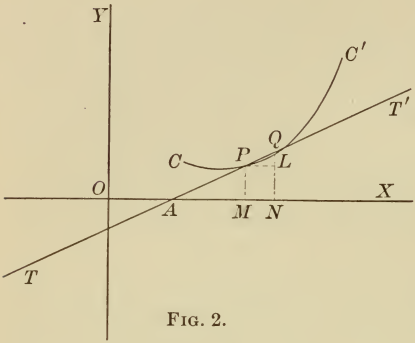

FIG. 2.

We now have a geometric meaning for the first derivative: *If $y = f(x)$, $\frac{dy}{dx}$ is the tangent of the angle which the tangent to the curve makes with the x-axis.*

The direction of the tangent determines the direction of the curve at the point of tangency. The value of $\frac{dy}{dx}$ at any particular point on the curve gives us, therefore, the **slope** or **gradient** of the curve at that point.

If $\alpha = \tan^{-1}\frac{dy}{dx}$, we also have, when $P$ and $Q$ are indefinitely near to each other, $\frac{LQ}{PQ} = \frac{dy}{ds} = \sin\alpha$, and $\frac{PL}{PQ} = \frac{dx}{ds} = \cos\alpha$, $ds$ being the elementary arc $PQ$.

**39.** The student will at once perceive that the first derivative must be of great use in searching for special features of any graph. For one important application, let us see what it can tell us about the graph of $ax + by + c = 0$. Differentiating this expression,

$$a + b\frac{dy}{dx} = 0,$$

and therefore

$$\frac{dy}{dx} = -\frac{a}{b}.$$

We have here a constant value for the tangent of the angle which the graph of $ax + by + c = 0$ makes with the $x$-axis. Accordingly, the slope is constant and the graph can have no bends; for a bend means change of slope. Therefore $ax + by + c = 0$ must be a *straight* line and its own tangent. But $ax + by + c = 0$ is the *general* equation of the first degree, and any property proved for it holds for any and every particular equation of the first degree. For instance, the graph of $y = x + 2$, which seemed in Art. 36 to be a straight line, we now know to be a straight line. Further, from $y = x + 2$ we have $\frac{dy}{dx} = 1$. Since 1 is the gradient of this particular line, we know that it makes an angle of 45° with the horizontal axis.

Again, in the curve $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$. Here $\tan \alpha$ varies inversely as the square of the abscissa, and is all the time negative. It follows that at every point the tangent to the curve makes an obtuse angle with the $x$-axis.

The angle $\alpha$ is always measured from the $x$-axis on the right-hand side of the origin, counter-clockwise around to the line which, with the $x$-axis, forms the angle.

### Exercises.

**40.** 1. A point moves in a circle around the origin as a center, with a radius $r$.

(1) The equation to the circle must be $x^2 + y^2 = r^2$; for the abscissa and ordinate are all the while the sides of a right triangle.

(2) Show that $$\frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}.$$

(3) Find the coördinates of the point or points where the circle has a slope of 1.

2. Find the point of tangency when the tangent to $y = \frac{1}{x}$ makes equal angles with the axes of reference.

Put $\frac{dy}{dx} = \tan 135°$ and solve for $x$.

3. Show that the curve $y = \frac{x}{1 + x^2}$ goes through the origin. Find its slope at the origin.

4. Construct the curve $y^2 = 4x$. Find the point of tangency when the tangent to the curve makes an angle of 45° with the $x$-axis.

5. Construct the curve $y = \sin x$, making as much use as possible of $\frac{dy}{dx}$ to determine the slope at various points.

The $x$-axis must here be regarded the circumference of a circle whose radius is unity, straightened to a right line with the origin marked 0°. We easily obtain a number of points on the curve by using the pairs of coördinates: 0°, 0; 45°, $\frac{1}{2}\sqrt{2}$; 90°, 1; 135°, $\frac{1}{2}\sqrt{2}$; 180°, 0; 225°, $-\frac{1}{2}\sqrt{2}$, etc. Hence the curve passes through the origin, has a maximum ordinate at 90°, and crosses the $x$-axis again at 180°. In order to measure off the abscissas, the angles 45°, 90°, 135°, etc., must be expressed in *radians*. We have the radian 57°.2958 ⋯ for the unit of distance. The abscissa indicated by 45°, for instance, is $\frac{45}{57.2958} = \frac{150}{191}$ approximately. The distance from the origin to the second point of crossing is $\frac{180}{57.2958} = 3.14159 \cdots$, and the maximum ordinate therefore meets the $x$-axis at a distance $\frac{1}{2}(3.14159 \cdots)$ from the origin.

From 180° to 360° the values of the sines are a repetition of the values for the first semi-circumference, except that they are now all negative. Hence this portion of the curve is in every respect like the portion from 0° to 180°; but it lies below the $x$-axis, and the direction of its convexity is reversed.

Since $\sin(n\pi + x) = \sin x$, $n$ being even and positive, it is seen that the curve keeps its sinuous character, crossing the $x$-axis at regular intervals an unlimited number of times. On account of the repetition over and over again of the series of values of $\sin x$, the function is called a **periodic function.** The curve itself is known as the **sinusoid.**

6. Construct the curve $y = \cos x$.

It is obvious in advance that this curve, which might be called the *co-sinusoid,* must be precisely like the sine curve or sinusoid; and that we shall have it in its proper position if we suppose the sine curve moved a distance of 90° to the left along the $x$-axis.

7. Find the first point to the right of the $y$-axis where $y = \sin x$ and $y = \cos x$ cross each other (see Art. 95). Show that the angle at which they cross is $180° - 2\tan^{-1}\frac{1}{2}\sqrt{2}$.

8. Construct $y = m \sin nx$.

9. Construct $y = m \cos nx$.

Assign numerical values to $m$ and $n$; then give a series of values to $x$, as in the first case. If a negative value is given to $m$, the effect is to rotate the curve on the $x$-axis so that portions which were above are now below, and *vice versa.*

**41.** If $\frac{dy}{dx} = 0$ for some value of $x$, $\alpha = 0$; hence, to find whether the point describing a curve is anywhere moving parallel to the $x$-axis, we must put $\frac{dy}{dx}$ equal to zero. Let $x_1$ represent one root of the equation $\frac{dy}{dx} = 0$. If a value of $x$ a little less than this root makes $\frac{dy}{dx}$ positive, and a value a little greater makes it negative, the tangent to the curve must make an acute angle with the $x$-axis, then become parallel to it, then make an obtuse angle with it; and the curve must have a bend, being convex upward. The ordinate of the highest point, corresponding to $x_1$, is a **maximum.** So we define a maximum value of a function as a value greater than the value just before it and also than the one just after it. (*PB*, Fig. 3.)

On the other hand, if $\frac{dy}{dx}$ changes from $-$ to $+$ in passing through zero, the curve is concave upward, and the lowest point is the end of a **minimum** ordinate; that is, the value of the function is less than the value just before it and the one just after it. ($P'C$, Fig. 3.)

It is evident that if $\frac{dy}{dx}$ is changing from + to −, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is −; and if $\frac{dy}{dx}$ is changing from − to +, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is +.

**42.** A third case arises: If $\frac{dy}{dx}$ does not change sign in passing through zero, there is neither a maximum nor a minimum; but the point after reaching $P$ or $P'$ takes the path indicated by the dotted line. The point where $\frac{dy}{dx} = 0$ is then called a **point of inflexion**. In this case $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0.$

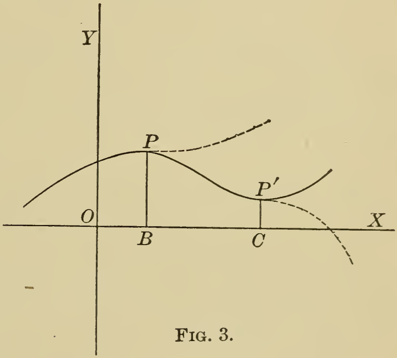


FIG. 3.

Every one is familiar with the point of inflexion as a feature in railroads, when the track is concave, say with respect to the fields on the right, and then changes so as to be concave to the fields on the left. Curves containing points of inflexion are very common in architectural forms. Such a curve is then known as an *ogee*.

The same curve may of course have several maximum points and several minimum points, and also points of inflexion. Maximum and minimum points must evidently alternate.

### Exercises.

**43.** 1. Consider the meaning of the statement $\frac{dy}{dx} = \infty$. Examine the two cases: (*a*) when $\frac{dy}{dx}$ changes sign in passing through an infinite value; (*b*) when $\frac{dy}{dx}$ does not change sign in passing through such a value.

2. Examine the following curves for maxima and minima:

(i) $y = \frac{x}{1+x^2}$.　　　(iii) $y = \log x$.

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.　　　(iv) $y = 2px$.

3. Draw the curve $y = e^x$, showing that it lies wholly above the $x$-axis, crosses the $y$-axis at an angle of 45°, and has no maximum or minimum points for any finite value of $x$.

**44.** To illustrate the use of the principles established in Art. 41, suppose we know the slant height $a$ of a right cone and wish to find the radius of its base when the volume is a maximum. Let $y$ be the volume and $x$ the base; then

$$y = \frac{\pi x^2}{3}\sqrt{a^2 - x^2}.$$

$x$ and $y$, being mutually dependent variables, must admit of graphical representation; the abscissa of the point tracing the curve or graph is the varying radius, and the ordinate is the varying volume. Hence, if we put $\frac{dy}{dx}$ equal to zero and solve the equation so formed, the value of $x$ obtained will be the radius which gives the maximum volume. Differentiating

$$y = \frac{\pi x^2}{3}\sqrt{a^2 - x^2},$$

$$\frac{dy}{dx} = \frac{\pi}{3}\left[2x\sqrt{a^2 - x^2} + x^2\left(\frac{-2x}{2\sqrt{a^2 - x^2}}\right)\right]$$

$$= \frac{\pi}{3}\left[\frac{2x(a^2 - x^2) - x^3}{\sqrt{a^2 - x^2}}\right];$$

putting this expression equal to zero,

$$2x(a^2 - x^2) - x^3 = 0;$$

hence $$x = \sqrt{\tfrac{2}{3}}\,a.$$

That is, the volume of the cone will be greatest when the radius of the base is $\sqrt{\frac{2}{3}}\,a$.

In a case like this it is unnecessary to inquire whether $\frac{dy}{dx}$ changes sign, and whether the change is from + to − or from − to +. For the volume of a cone of given slant height evidently varies from no volume when the radius is zero, through finite values to no volume again, when the radius is equal to the slant height; that is, from a cone that is all height and no base to one that is all base and no height. Somewhere between these two extreme cases there must be a cone of ordinary shape whose volume is the greatest possible. It is well occasionally to supplement mathematics with common sense rather than to rely mechanically and invariably on some rule or formula.

**Examples.**

1. Find the altitude of the right cylinder of greatest volume inscribed in a sphere whose radius is $r$.

$$\text{Alt.} = \frac{2r}{\sqrt{3}}.$$

2. Given a point on the axis of the parabola $y^2 = 4px$, at the distance $l$ from the vertex, find the abscissa of the point of the curve nearest to it. $x = l - 2p$.

3. Find the maximum rectangle that can be inscribed in the ellipse whose axes are $a$ and $b$.

The sides are $a\sqrt{2}$ and $b\sqrt{2}$.

4. A talus resting on a horizontal plane has a slope of 30°; at the top of the talus is a series of strata 5 ft. thick; the entire height of the ledge is 30 ft. How far must one stand from the foot of the talus to get the best view of the strata?

The angle at the observer's eye, formed by lines drawn to the bottom and to the top of the strata, must be a maximum. Let this angle be $\alpha$; the angle subtended by the talus, $\beta$, and the angle subtended by both talus and strata, $\gamma$. Also, let $x$ be the horizontal distance from the observer to a point directly beneath the strata. Then $\tan\beta = \frac{25}{x}$; $\tan\gamma = \frac{30}{x}$; and

therefore $$\tan\alpha = \frac{\frac{30}{x} - \frac{25}{x}}{1 + \frac{750}{x^2}} = \frac{5x}{x^2 + 750};$$

and $$\frac{d}{dx}\tan\alpha = \frac{5(x^2 + 750) - 10x^2}{(x^2 + 750)^2}.$$

Equating this derivative to zero, $x = 5\sqrt{30}$, and finally the distance sought is

$$5\sqrt{30} - \frac{25}{\tan 30^\circ} = 5\sqrt{3}(\sqrt{10} - 5).$$

5. The strength of a rectangular beam of given length, loaded and supported in any particular way, is proportional to the breadth of the section multiplied by the square of the depth. If the diameter $a$ is given of a cylindric tree, what is the strongest beam which may be cut from it?

Let $x$ be the beam's breadth; then $\sqrt{a^2 - x^2}$ must be its depth. Hence, if $y = x(a^2 - x^2)$, the strength is a maximum when $y$ is a maximum.

$$\frac{dy}{dx} = a^2 - 3x^2 = 0, \text{ and therefore } x = \frac{a}{\sqrt{3}}.$$

In the same way find the *stiffest* beam which may be cut from the tree by making the breadth multiplied by the cube of the depth a maximum. We now have

$$y = x(a^2 - x^2)^{\frac{3}{2}};$$

$$\frac{dy}{dx} = (a^2 - x^2)^{\frac{3}{2}} + \tfrac{3}{2}x(a^2 - x^2)^{\frac{1}{2}}(-2x) = 0, \text{ and } x = \frac{a}{2}.$$

— Perry's *Calculus for Engineers.*

6. The volume of a circular cylindric cistern being given (no cover), show that its surface is a minimum when the radius of the base is equal to the height of the cistern.

Let $x$ be the radius and $y$ the height; then the volume is $\pi x^2 y$, which equals a constant, say $a$. If $S$ is the surface,

$$S = \pi x^2 + 2\pi xy = \pi x^2 + \frac{2a}{x}, \text{ since } y = \frac{a}{\pi x^2}.$$

Finding $\frac{dS}{dx}$ and putting it equal to zero, we have $x^3 = \frac{a}{\pi} = \frac{\pi x^2 y}{\pi}$, and $x = y$. How do we know that this makes the surface a minimum rather than a maximum?

7. Determine the speed most economical in fuel to steam against a tide, supposing the resistance to vary as the $n$th power of the velocity through the water.

Let $a$ denote the velocity of the tide, $x$ the velocity of the steamer through the water; then $x - a$ will be the velocity of the steamer relatively to the bank. The power required, and therefore the coal burnt per hour, will vary as the product of the resistance and the speed; that is, as $x^{n+1}$, and therefore the coal burnt per mile will vary as $\frac{x^{n+1}}{x-a}$. This is to be a minimum, hence we have

$$\frac{d}{dx}\left(\frac{x^{n+1}}{x-a}\right) = \frac{(n+1)x^n(x-a) - x^{n+1}}{(x-a)^2} = 0;$$

and $$\frac{x}{a} = 1 + \frac{1}{n}, \text{ or } \frac{x-a}{a} = \frac{1}{n}.$$

Thus if the resistance is taken to vary as the square of the velocity, the speed past the bank should be half the velocity of the current.

— Greenhill's *Differential and Integral Calculus.*

8. Let $A$ and $B$ be two point-sources of heat. It is required to find the point $M$ on the straight line $AB$, which is at the lowest temperature, the intensity of the radiation of heat varying inversely as the square of the distance from the source of heat. Let $a$ be the distance

between the points $A$ and $B$, and $x$ the distance from $A$ of the point $M$ on the straight line; then

$$AM = x, \text{ and } BM = a - x.$$

Let the intensities of heat at unit distance from the sources of heat be denoted by $\alpha$ and $\beta$ respectively. Then the total intensity of heat $\omega$ at the point $M$ will be

$$\omega = \frac{\alpha}{x^2} + \frac{\beta}{(a-x)^2}.$$

For a maximum or minimum,

$$\frac{d\omega}{dx} = -\frac{2\,\alpha}{x^3} + \frac{2\,\beta}{(a-x)^3} = 0,$$

that is,
$$\frac{(a-x)^3}{x^3} = \frac{\beta}{\alpha},$$

and
$$\frac{a-x}{x} = \frac{\sqrt[3]{\beta}}{\sqrt[3]{\alpha}}.$$

The distances $BM$ and $AM$ have, therefore, the same ratio as the cube roots of the corresponding heat intensities.

Solving for $x$,

$$x = \frac{a\sqrt[3]{\alpha}}{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}.$$

In this case it is necessary to see whether the value found corresponds to a maximum or a minimum. Differentiating the expression for $\frac{d\omega}{dx}$, we have

$$\frac{d^2\omega}{dx^2} = \frac{2 \cdot 3\,\alpha}{x^4} + \frac{2 \cdot 3\,\beta}{(a-x)^4};$$

which is positive for all values of $x$, including the value

$$\frac{a\sqrt[3]{\alpha}}{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}.$$

$\omega$ is therefore a minimum.*

**45.** Suppose a line $AB$ through the origin to revolve around $O$ counter-clockwise, making the variable angle $\theta$ with the $x$-axis. Let $AB$ pass through a point $P(x, y)$, and let the distance of $P$ from $O$ be denoted by $r$. $r$ and $\theta$ are called **polar coördinates**; $O$ is the **pole**.

Projecting $OP$ onto the $x$ and $y$ axes respectively, we have $x = r\cos\theta$ and $y = r\sin\theta$. Through these relations $F(x, y) = 0$ becomes $F(r\cos\theta, r\sin\theta) = 0$. For example, the equation $x^2 + y^2 - 2ay = 0$ becomes in polar coördinates $r^2\cos^2\theta + r^2\sin^2\theta - 2a(r\sin\theta) = 0$; that is, $r = 2a\sin\theta$. This is readily seen to be a circle to which the $x$-axis is tangent, the point of tangency being the origin, or pole.

Whenever any value of $\theta$ makes $r$ negative, we measure from the origin away from that end of the line $AB$ which is tracing the arc that measures $\theta$. If we imagine an arrow to lie in the line $AB$ and rotate with it, the barb may be regarded as tracing the arc that measures $\theta$, while the feather-end is negative.

**46.** Let $PP'$ be an arc $\delta s$ of a curve $f(r, \theta) = 0$, $PQ$ the arc of a circle whose radius is $r$; and let the angle $POP' = \delta\theta$. In the limit $PQP'$ is a right triangle, $PQ = rd\theta$, $QP' = dr$, and $PP' = ds$. Let $\phi$ be the

* Nernst and Schönflies' *Einführung in die mathematische Behandlung der Naturwissenschaften.*

angle made by the **radius vector** $OP$ and the tangent to the curve; then $\tan\phi = \frac{PQ}{QP'} = \frac{rd\theta}{dr}$.

Whenever the radius vector $r$ is a maximum or minimum, the tangent at its extremity must be at right angles to it; that is,

$$\frac{rd\theta}{dr} = \infty \text{ or } \frac{dr}{rd\theta} = 0.$$

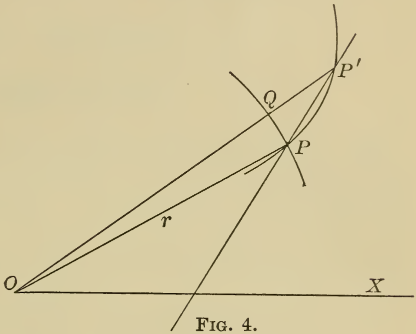

FIG. 4.

Points for which $r$ is a maximum or minimum are called **apsides**. To find, therefore, whether a given curve has an apsis, we must put $\frac{1}{r}\frac{dr}{d\theta} = 0$ and solve this equation.

For example, let us take the polar equation to the ellipse, the pole being at the right hand focus (see Art. 115).

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \text{ and } \frac{dr}{d\theta} = \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2};$$

then $\frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}$; and equating this to zero, $\sin\theta = 0$. Hence, the apsidal values of $\theta$ are 0° and 180°. These results agree with what we observe in an examination of the given equation to the ellipse: $r$ is a maximum, $a(1+e)$, when $\theta = 180°$, and a minimum, $a(1-e)$, when $\theta = 0°$.

The student who is unacquainted with the formal analytic geometry of the straight line and the conic section is advised to read Chapter IV before beginning the next chapter.

# CHAPTER III.

## APPLICATIONS.

**47.** In the mathematical sciences one of the most common of fundamental variables is *time;* and when the function of time is the *space* passed over by a body, the first and second derivatives $\frac{ds}{dt}$ and $\frac{d}{dt}\left(\frac{ds}{dt}\right)$ are of great importance.

Suppose a body moves over equal spaces in equal times. The space divided by the time gives the speed or **velocity** of the body. That is, if $s$ is the space passed over in the time $t$, $\frac{s}{t}$ is the velocity of the body.

"While the camels were being loaded, I measured my first base-line of 400 metres. Boghra (my riding camel) walked it in five and one-half minutes. This was a daily recurring task, for the contours of the ground varied a good deal, and the depth of the sand made a very appreciable difference in the time the camels took to do the same distance."

—Sven Hedin's *Through Asia*, Vol. I, p. 482.

In this illustrative case, $\frac{s}{t} = \frac{400}{5\frac{1}{2}}$ = the speed of the camel expressed in metres per minute. Assuming that $\frac{s}{t}$ was a constant during each day, the distance travelled on any given day by Hedin's caravan was known by multiplying the speed by the time spent in travel.

**48.** If the motion is variable so that the body does not move over equal spaces in equal times, we may obtain an expression for velocity by taking the time so short that during that time the motion must be uniform. So if $dt$ be an indefinitely short time and $ds$ the indefinitely small space passed over in that time, $\frac{ds}{dt}$ is the velocity and is measured by the space that would have been passed over in a unit of time if the body had kept on moving for a whole unit with the velocity which it had at the instant considered. For instance, if we say that a train is running at the rate of 30 miles an hour, we mean that if it were to run for a whole hour with the same speed which it has at this instant it would pass over a distance of 30 miles. As a matter of fact it may stop in a few minutes; that has nothing to do with its speed at this instant. But 30 miles per hour is the same as 1 mile in 2 minutes, or 4.4 feet in .1 of a second, and so on. Evidently the *rate* remains the same so long as the ratio of the space to the time is the same, however small the space and the time may be individually. Hence, in this case, $\frac{ds}{dt} = 30$ miles per hour.

If we know the whole space passed over by a body and know also the time taken, the space divided by the time is the *average* velocity: it must not be confused with the velocity proper, which may have varied during the time. For example, the first mail cartridge sent by compressed air from the Boston post office to the North Union Station (Dec. 17, 1897) required 1 minute and 2 seconds to pass from one place to the other, a distance of 4500 feet. The average velocity was 72.58+ feet per second.

**49.** If a body is moving in a northeasterly direction, it plainly has a motion eastward and a motion northward. For instance, if it is moving due northeast with a velocity of 20 miles per hour, it is getting eastward at the rate of 20 cos 45° miles per hour, and northward at the same rate. If it is moving east 30° north at the rate of 20 miles per hour, it is moving east at the rate of 20 cos 30° miles per hour, and north at the rate of 20 cos 60° miles per hour.

In general, if a body is moving with a velocity $v$ along a line which makes with the $x$-axis an angle of $\alpha$ degrees, its **component velocity** parallel to the $x$-axis is $v \cos \alpha$, and its component velocity parallel to the $y$-axis is $v \sin \alpha$. We have already seen (Art. 38) that $\frac{dx}{ds} = \cos \alpha$ and $\frac{dy}{ds} = \sin \alpha$. Hence, if $\frac{ds}{dt}$ is the velocity of a body at any instant, $v \cos \alpha = \frac{ds}{dt} \cdot \frac{dx}{ds} = \frac{dx}{dt}$; and $\frac{dx}{dt}$ is therefore the component velocity parallel to the $x$-axis. Similarly, $v \sin \alpha = \frac{ds}{dt} \cdot \frac{dy}{ds} = \frac{dy}{dt} =$ the component velocity parallel to the $y$-axis.

Evidently a velocity parallel to any line furnishes a component velocity parallel to any other line if it be multiplied by the cosine of the angle between the lines.

**50.** Suppose a particle is moving in a plane curve and we wish to know its component velocities at any instant (1) along the radius vector, and (2) perpendicular to the radius vector.

We have $x = r \cos \theta$ and $y = r \sin \theta$, in which $x$, $y$, $r$, and $\theta$ depend upon the time $t$. Differentiating with $t$ as the fundamental variable,

$$\frac{dx}{dt} = \frac{dr}{dt} \cos\theta - r \sin\theta \frac{d\theta}{dt}, \qquad (a)$$

$$\frac{dy}{dt} = \frac{dr}{dt} \sin\theta + r \cos\theta \frac{d\theta}{dt}. \qquad (b)$$

According to the preceding article, $\frac{dx}{dt}$ is the velocity parallel to the $x$-axis and $\frac{dx}{dt} \cos\theta$ is the component which it furnishes along the radius vector. Similarly, $\frac{dy}{dt} \sin\theta$ is the component which $\frac{dy}{dt}$ furnishes along the radius vector. The sum of these components is the

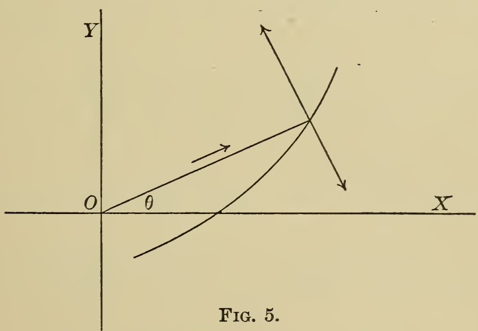


FIG. 5.

whole velocity along the radius vector. From equations ($a$) and ($b$) we have

$$\frac{dx}{dt} \cos\theta + \frac{dy}{dt} \sin\theta = \frac{dr}{dt}. \qquad (c)$$

Again, resolving along a line perpendicular to the radius vector and combining the parts,

$$\frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta = r\frac{d\theta}{dt}. \qquad (d)$$

The reason for the minus sign in the first member of equation ($d$) should be noticed. The velocities $\frac{dy}{dt}\cos\theta$ and $\frac{dx}{dt}\sin\theta$ are oppositely directed (see Fig. 5); hence, when combined, their difference must be expressed.

**51.** $\frac{d}{dt}\left(\frac{ds}{dt}\right)$, the rate of change of a variable velocity, is called **acceleration.**

$\frac{d}{dt}\left(\frac{dx}{dt}\right)$ and $\frac{d}{dt}\left(\frac{dy}{dt}\right)$ are the accelerations parallel to the $x$-axis and $y$-axis respectively; and we can now find the component accelerations (1) along the radius vector, and (2) perpendicular to the radius vector.

Differentiating equations ($a$) and ($b$) of the preceding article,

$$\frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\cos\theta - \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\sin\theta, \qquad (e)$$

$$\frac{d^2y}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta - \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\cos\theta. \qquad (f)$$

Multiplying equation ($f$) by $\sin\theta$, and equation ($e$) by $\cos\theta$ and adding, we have

$$\frac{d^2y}{dt^2}\sin\theta + \frac{d^2x}{dt^2}\cos\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad (g)$$

for the acceleration along the radius vector.

Again, multiplying ($f$) by $\cos\theta$, and ($e$) by $\sin\theta$ and subtracting the latter product from the former, we have

$$\frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \qquad (h)$$

for the acceleration perpendicular to the radius vector.

It is to be noticed that the second member of equation ($h$) may be written $\frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$.

**52. Angular velocity** is defined as the ratio of the angle differential, $d\theta$, to the time differential, $dt$. This ratio, $\frac{d\theta}{dt}$, may be a constant or a variable. For example, the earth rotates on her axis with constant angular velocity, and $\frac{d\theta}{dt} = \frac{360^\circ}{24^h}$; but she moves in her orbit around the sun with a variable angular velocity. (See Art. 88.)

**53.** As an important application of the results given in equations ($g$) and ($h$) above, suppose a particle is moving in a circle with constant angular velocity. Then, since $r$ is a constant, $\frac{dr}{dt} = 0$ and $\frac{d}{dt}\left(\frac{dr}{dt}\right) = 0$. Therefore the acceleration along the radius vector reduces to $-r\left(\frac{d\theta}{dt}\right)^2$. Also, in equation ($h$), since $\frac{dr}{dt} = 0$, the term $2\frac{dr}{dt}\frac{d\theta}{dt}$ is zero; and since $\frac{d\theta}{dt}$ is a constant, $\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = 0$, and the term $r\frac{d^2\theta}{dt^2}$ is zero; and therefore the acceleration perpendicular to the radius vector is zero. This conclusion is what we might have expected from the premise that the particle moves in a circle with constant angular velocity.

**54.** The above expression, $-r\left(\frac{d\theta}{dt}\right)^2$, for the acceleration along the radius vector when a particle is moving in a circle with constant angular velocity, may be written

$$-\frac{r^2}{r}\left(\frac{d\theta}{dt}\right)^2 \quad \text{or} \quad -\frac{1}{r}\left(\frac{rd\theta}{dt}\right)^2;$$

but since $rd\theta$ is the length of the arc corresponding to the angle $d\theta$, $\frac{rd\theta}{dt}$ is the *linear* velocity $v$ of the particle. Hence we have $-\frac{v^2}{r}$ as a simple form for the acceleration along the radius vector when the particle or body moves in a circle with constant angular velocity.

**55.** Suppose a point $Q$ moves with constant angular velocity $\frac{d\theta}{dt}$ or $\omega$ in a circle $AQA'$ whose radius is $r$. Take the center as origin, and let $QP$ be the perpendicular from $Q$ to the $y$-axis $OA$.

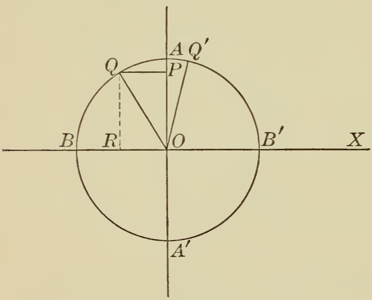

FIG. 6.

As the angle $AOQ$ increases, the line $QP$ increases from 0 to $r$ and then decreases. The changes in the ratios $\frac{QP}{OQ}$, $\frac{OP}{OQ}$, etc., due to changes in the angle $AOQ$, have already been discussed (Art. 20). We shall now consider the motion of the point $P$ as

$Q$ describes the circle. $OP$ is the ordinate of $P$ at any instant, and if $Q$ has taken the time $t$ to move from $A$ to $Q$, the angle $AOQ = \omega t$; hence $y = r \cos \omega t$.

If $Q$ starts at some point $Q'$, and $t_0$ is the time required to move from $Q'$ to $A$, the angle $Q'OA = \omega t_0$; hence, counting the time from the start at $Q'$, the angle $Q'OQ = \omega t$ and the angle $AOQ = \omega t - \omega t_0$; and therefore

$$y = r \cos(\omega t - \omega t_0) = r \cos(\omega t + \epsilon)$$

if we write $\epsilon$ for the constant, $-\omega t_0$.

**56.** In regard to the motion of $P$, we notice at once that it must cross the circle on the diameter $AA'$ and return to $A$ in the time that $Q$ is describing the circumference; so its motion is vibratory. It starts with zero velocity, and must be going with its greatest velocity when at the center; for its direction of motion is then parallel to that of $Q$.

To get a more precise knowledge of the motion of $P$, let us take $\epsilon = 0$, so that $y = r \cos \omega t$. By doing this the equation gains in simplicity and the motion remains the same, but the time is counted from the instant when $Q$ is at $A$ instead of $Q'$.

We now have $y = r \cos \omega t,$ (a)

$$\frac{dy}{dt} = -r\omega \sin \omega t, \qquad (b)$$

$$\frac{d^2y}{dt^2} = -r\omega^2 \cos \omega t. \qquad (c)$$

Equation (b) shows that the velocity of the point is greatest when $\omega t = 90°$; that is, when $P$ is at the center. Equation ($c$) shows that the acceleration is greatest when $\omega t = 0°$ and $180°$; that is, at the start and at $A'$. Also, the acceleration is least when $\omega t = 90°$.

**57.** The variation in the ordinate $OP$ may be best appreciated by noticing the identity of the equation $y = r \cos \omega t$ with the equation $y = m \cos nx$ given in exercise 9, Art. 40. Equation ($a$) accordingly represents a cosine curve. Further, if the velocity equation ($b$) be graphically shown, its curve must be the sinusoid. And finally, the acceleration equation ($c$) is another cosine curve, differing from the first, however, on account of the coefficient $-r\omega^2$, which has replaced the coefficient $r$ in equation ($a$).

It is well worth the student's while to construct carefully the graphs for the three equations ($a$), ($b$), ($c$), using the same unit of length for all three. The usual $x$-axis now becomes a *time* axis in each case, since the abscissas are times. The $y$-axis for ($a$) is a *displacement* axis; for ($b$) it is a *velocity* axis; and for ($c$) an *acceleration* axis.

**58.** We are now familiar with the geometrical meaning of $\frac{dy}{dx}$ when $y = f(x)$. If $y = f(t)$, $\frac{dy}{dt}$ is analogous to $\frac{dy}{dx}$, and must have the same geometrical meaning. That is, viewed geometrically rather than kinematically, $\frac{dy}{dt}$ is the tangent of the angle which the tangent to the curve $y = f(t)$ makes with the $t$-axis. Accordingly, equation ($b$), Art. 56, might be called the curve of the tangent to ($a$); for any ordinate (with the abscissa $t'$)

in the graph of $(b)$ represents the magnitude of the slope of $(a)$ at the point whose abscissa is $t'$. Evidently the curve of $(c)$ is related to $(b)$ just as $(b)$ is to $(a)$.

**59.** The point $P$, vibrating back and forth across the circle (Fig. 6), is said to have **simple harmonic motion.** It is such motion as this that Jupiter's satellites seem to have as we look at his orbit "edge on."

The range $OA$ or $OA'$ on one side or the other of the middle point is called the **amplitude**; and the ordinate $OP$ is the **displacement.** The **period** of a simple harmonic motion is the time which elapses from any instant until the point moves again in the same direction through the same position; that is, the time required by $P$ to move from $P'$ to $A'$, thence back to $A$, and finally to the initial position $P'$, is the period. The **phase** is the fraction $\frac{\omega t}{2\pi}$ of the period of vibration. The **epoch** is the angle $\epsilon$.

"This expression $y = r \cos(\omega t + \epsilon)$ is to be found, perhaps more frequently than any other, in all branches of mathematical physics. It is in terms, or series of terms, of this form that *every* periodic phenomenon can be described mathematically. From the expressions for the longitude and radius vector of a planet or satellite to those of the most complex undulations, whether in water, in air, or in the luminiferous medium, all are alike dependent upon it."

—Tait's *Dynamics.*

*Example.* Find an expression for the up and down motion of the connecting-rod of a locomotive.

**60.** The downward fall of an unsupported body is due to the accelerating force exerted by the earth and known as **gravity.** At small distances above the earth's surface this force is practically constant; the acceleration caused by it is denoted by $g$. When $g$ is determined at different places on the earth, it is found to vary within narrow limits. This variation is due to several causes, the chief one being the rotation of the earth on its axis. $g$ has its least value at the equator and its greatest value at the poles. At Washington, D.C., $g$ is 980.098 dynes;* that is, the observed acceleration due to gravity is, at that point on the earth's surface, 32.155 feet per second.

Taking the origin at the point from which a body falls, with the positive end of the $y$-axis downward, we now have

$$\frac{d}{dt}\left(\frac{dy}{dt}\right)=g,$$

$$d\left(\frac{dy}{dt}\right)=g\,dt;$$

therefore, after integrating,

$$\frac{dy}{dt}=v=gt+C. \qquad \text{(Art. 15.)}$$

If the body falls from rest, $v=0$ when $t=0$; therefore $C=0$, and the equation becomes

$$\frac{dy}{dt}=gt. \qquad (a)$$

Multiplying by $dt$ and integrating again,

$$y=\tfrac{1}{2}gt^2+C'.$$

Since $y=0$ when $t=0$, $C'=0$;

therefore $$y=\tfrac{1}{2}gt^2. \qquad (b)$$

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*U. S. Coast and Geodetic Survey.

Combining equations ($a$) and ($b$) so as to eliminate $t$,

$$\frac{dy}{dt} = v = \sqrt{2gy}. \qquad (c)$$

Equation ($c$) enables us to find the velocity with which a body is moving when it has fallen through a given space. For example, the monument at Washington is 555 feet high; if a ball is dropped from the top, what is its velocity upon reaching the ground? We may take $g = 32$, a value sufficiently accurate in this example and similar ones. Then $v = 8\sqrt{555} = 188$ feet per second, approximately.

**61.** If the body is projected directly upward,

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = -g,$$

because the acceleration is now a retardation tending to diminish $y$. Integrating as before,

$$\frac{dy}{dt} = -gt + C.$$

If the body is projected with the velocity $V$, $\frac{dy}{dt} = V$ when $t = 0$; therefore $C = V$, and the equation becomes

$$\frac{dy}{dt} = -gt + V. \qquad (d)$$

Multiplying by $dt$ and integrating again,

$$y = -\tfrac{1}{2}gt^2 + Vt + C',$$

and since $y = 0$ when $t = 0$, $C' = 0$; and we have

$$y = -\tfrac{1}{2}gt^2 + Vt. \qquad (e)$$

Combining equations ($d$) and ($e$) so as to eliminate $t$,

$$v^2 = V^2 - 2gy. \qquad (f)$$

**62.** By means of the equations of the two preceding articles we can readily show that if a body is projected vertically upward, it takes the same time to come down that it does to go up; also, upon reaching the point from which it was projected, it has the same velocity as that with which it was projected.

From ($d$), when $\dfrac{dy}{dt} = 0, \quad t = \dfrac{V}{g}$ = time up;

from ($f$), when $\dfrac{dy}{dt} = 0, \quad y = \dfrac{V^2}{2g}$ = space up;

from ($b$), when $y = \dfrac{V^2}{2g}, \quad t = \dfrac{V}{g}$ = time down;

from ($c$), when $y = \dfrac{V^2}{2g}, \quad v = V.$

In the derivation of formulas ($a$) to ($f$), no account has been taken of the resistance offered by the air to the fall or rise of a body. The formulas are strictly true only on the supposition that the acceleration is constant, and that the motion takes place in a vacuum.

**63.** We may now consider the case when the height is so great that the acceleration cannot be regarded as constant. What is the velocity of the body on reaching the earth?

A homogeneous sphere, or a sphere composed of concentric layers with the density varying only from one layer to another, attracts an external body with an intensity varying inversely as the square of the distance of the body from the center of the sphere. Let $g$ be the acceleration due to the earth when the body is at the earth's surface, and $f$ the acceleration at the distance $y$ from the center. (Notice that the center thus becomes the origin.)

Then, if $R$ is the earth's radius,

$$\frac{f}{g}=\frac{R^2}{y^2}; \text{ that is, } f=\frac{gR^2}{y^2},$$

and therefore we have

$$\frac{d}{dt}\left(\frac{dy}{dt}\right)=-\frac{gR^2}{y^2}.$$

The minus sign is taken because $y$ is diminishing as the body falls; that is, $dy$ is negative, and since $\frac{dy}{dt}$ is increasing numerically, $\frac{d}{dt}\left(\frac{dy}{dt}\right)$ must also be negative.

If we multiply by $dt$, as in the previous articles, and attempt to integrate, we have

$$\frac{dy}{dt}=\int-\frac{gR^2}{y^2}\,dt,$$

an indicated operation which cannot be performed unless we know what function $y$ is of $t$; and this we do not know in advance. But multiplying by $dy$ instead of $dt$,

$$dy\frac{d}{dt}\left(\frac{dy}{dt}\right)=\frac{dy}{dt}d\left(\frac{dy}{dt}\right)=-\frac{gR^2}{y^2}\,dy.$$

The first member is immediately integrated by observing that it is of the form $xdx$, and that

$$\int xdx=\tfrac{1}{2}x^2.$$

We have, therefore,

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 = \int -\frac{gR^2}{y^2}\,dy = -gR^2\int\frac{dy}{y^2}$$

$$= -gR^2\left(-\frac{1}{y}\right) + C.$$

If the body falls from the height $h$ above the earth's surface so that $y = R + h$ when $\frac{dy}{dt} = 0$, $C = -\frac{gR^2}{R+h}$, and the equation becomes

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 = gR^2\left(\frac{1}{y} - \frac{1}{R+h}\right).$$

The same result is reached by writing a definite integral (Art. 16) whose limits are $R + h$ and $y$. We then have

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 = \int_{R+h}^{y} -\frac{gR^2}{y^2}\,dy$$

$$= -gR^2\left[-\frac{1}{y}\right]_{R+h}^{y}$$

$$= -gR^2\left(-\frac{1}{y} + \frac{1}{R+h}\right)$$

$$= \quad gR^2\left(\frac{1}{y} - \frac{1}{R+h}\right).$$

Suppose that $\frac{dy}{dt} = v'$ when $y = R$; that is, $v'$ is the velocity which the body has when it reaches the earth's surface. Then

$$\tfrac{1}{2}(v')^2 = gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right) = gR\left(\frac{h}{R+h}\right) = gh\left(\frac{R}{R+h}\right);$$

and therefore

$$v' = \sqrt{2gh\left(1 - \frac{h}{R} + \frac{h^2}{R^2} - \cdots\right)}.$$

If $h < R$, the series within the parenthesis is converging; and if $h$ is very small in comparison with $R$, we may drop all terms of the series after the first term; we then have $v' = \sqrt{2gh}$, which is identical with formula ($c$), Art. 60.

If $h > R$, the series is diverging; the formula containing it cannot therefore be used, and we return to one of the other expressions. For example, suppose a body falls from an indefinitely great distance; what will be its velocity on reaching the surface of the earth, all forces besides the earth's attraction being disregarded?

We have $$\tfrac{1}{2}(v')^2 = gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right),$$

or $$v' = \sqrt{2gR},$$

when $h$ is indefinitely great.

If $R = 3960 \times 5280$ feet and $g = 32.155$,

$$v' = \frac{\sqrt{2(32.155)(3960)(5280)}}{5280} \text{ miles per second,}$$

$$= 7 \text{ miles per second, approximately.}$$

By "great heights" we may mean such various heights as those attained by the kites flown at the Blue Hill Observatory (8000 ft.), or by Andrée's balloon; or the height of a meteorite when it first becomes visible. In the practical consideration of the velocities of bodies falling from such heights, the resistance of the air must be taken into account. For a discussion of the vertical motion of a body in a resisting medium, see Greenhill's *Calculus*, Art. 76.

**64.** Let a body free to move be subjected to an attractive force that varies directly as the distance of the body from the point where the force is located. If we take this point as origin, with a line passing through the body for the $x$-axis, we have

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\mu x.$$

The coefficient $\mu$ is seen to be the value of the acceleration when $x=1$; that is, when the body is at a unit's distance from the origin. The minus sign is used for the same reason that was given in Art. 63.

Multiplying by $dx$ and integrating,

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = -\frac{\mu}{2}x^2 + C.$$

If $\dfrac{dx}{dt} = 0$ when $x = a$, $C = \dfrac{\mu a^2}{2}$;

therefore $$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{2}(a^2 - x^2).$$

Writing this equation so that $dt$ shall stand by itself,

$$dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2 - x^2}}.$$

After extracting the square root only the negative sign is retained; because $dt$ is positive, and $dx$ is negative when the body is moving toward the origin.

Integrating again,

$$t = \frac{1}{\sqrt{\mu}} \cos^{-1}\frac{x}{a} + C'.$$

If $t = 0$ when $x = a$, $C' = 0$;

therefore
$$t = \frac{1}{\sqrt{\mu}} \cos^{-1}\frac{x}{a};$$

that is,
$$x = a \cos\sqrt{\mu}t.$$

Comparing this result with equation ($a$), Art. 56, we conclude that a body subjected to an attractive force varying directly as the distance will move with simple harmonic motion.

**65.** Suppose the body is driven away from the origin by a force varying directly as the distance of the body.

Then
$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \mu x;$$

and proceeding as before,

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 + C = \mu(x^2 - a^2); \qquad (a)$$

that is,
$$\sqrt{\mu}dt = \frac{dx}{\sqrt{x^2 - a^2}}.$$

Integrating again, we have

$$t\sqrt{\mu} + C' = \log(x + \sqrt{x^2 - a^2}).$$

Notice that the constant of integration may be written in either member of the equation as suits our convenience. Heretofore it has been written in the right-hand member.

Now suppose that $x = a$ when $t = 0$;

then $$C' = \log a,$$

and $$t\sqrt{\mu} + \log a = \log(x + \sqrt{x^2 - a^2}),$$

$$t\sqrt{\mu} = \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right).$$

From this expression we have

$$x + \sqrt{x^2 - a^2} = ae^{\sqrt{\mu}t}.$$

Further, since $(x + \sqrt{x^2 - a^2})(x - \sqrt{x^2 - a^2}) = a^2$,

$$x - \sqrt{x^2 - a^2} = \frac{a^2}{ae^{\sqrt{\mu}t}} = ae^{-\sqrt{\mu}t}.$$

Adding the expressions for

$$x + \sqrt{x^2 - a^2} \text{ and } x - \sqrt{x^2 - a^2},$$

$$2x = ae^{\sqrt{\mu}t} + ae^{-\sqrt{\mu}t};$$

that is, $$x = \frac{a}{2}(e^{\sqrt{\mu}t} + e^{-\sqrt{\mu}t}). \qquad (b)$$

If we now differentiate this expression, we shall have the velocity a function of the time instead of a function of the distance as in equation ($a$); for

$$\frac{dx}{dt} = \frac{a}{2}(\sqrt{\mu}e^{\sqrt{\mu}t} - \sqrt{\mu}e^{-\sqrt{\mu}t})$$

$$= \frac{a\sqrt{\mu}}{2}(e^{\sqrt{\mu}t} - e^{-\sqrt{\mu}t}). \qquad (c)$$

Equations ($b$) and ($c$) show that as $t$ increases, the body is driven farther and farther from the origin with ever increasing velocity. These equations involve the supposition that the initial velocity is zero. Let us now suppose that the initial velocity is $-a\sqrt{\mu}$. Resuming the equation

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 + C,$$

since $\left(\frac{dx}{dt}\right) = -a\sqrt{\mu}$ when $x = a$, $C = 0$;

and the equation becomes

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2,$$

or

$$\frac{dx}{x} = -\sqrt{\mu}dt,$$

the minus sign being used because the motion is toward the origin.

We now have $\log x = -\sqrt{\mu}t + C'$,

and since $x = a$ when $t = 0$, $C' = \log a$;

therefore $-\sqrt{\mu}t = \log\frac{x}{a}$,

and $x = ae^{-\sqrt{\mu}t}$.

This equation shows that with the initial velocity $-a\sqrt{\mu}$ the body constantly approaches the origin, but never reaches it.

**66.** Suppose that a body instead of being projected vertically, is projected in a direction making the angle $\alpha$ with the horizontal plane, $V$ being the velocity of projection. The body thus has a vertical velocity and a horizontal velocity. The horizontal velocity is evidently unaccelerated, whilst the vertical velocity is being retarded by gravity. That is, taking the horizontal side of the angle $\alpha$ for the $x$-axis, and taking the $y$-axis vertical and positive upward with the point from which the body is projected as origin,

$$\frac{d}{dt}\left(\frac{dx}{dt}\right)=0; \quad \frac{d}{dt}\left(\frac{dy}{dt}\right)=-g.$$

These two statements are the "equations of motion" of the body. Examples of such equations have already occurred in preceding articles. Integrating the first one, $\frac{dx}{dt}=V\cos\alpha$, the constant horizontal velocity. Integrating again,

$$x = tV\cos\alpha, \tag{a}$$

the constant of integration being zero if $t=0$ when $x=0$.

Integrating the second equation of motion,

$$\frac{dy}{dt}=-gt+C.$$

When $t=0$, the time of projection, $\frac{dy}{dt}$ is the vertical component of the velocity for the same instant. This initial vertical velocity being $V\sin\alpha$, we have

$$\frac{dy}{dt}=-gt+V\sin\alpha,$$

and integrating again,

$$y=-\tfrac{1}{2}gt^2+tV\sin\alpha. \tag{b}$$

Equations ($a$) and ($b$) give the coördinates of the body at any time $t$. Eliminating $t$, we have

$$y=x\tan\alpha-\frac{g}{2\,V^2\cos^2\alpha}x^2, \tag{c}$$

the equation to the path of the body.

**67.** If we transform equation ($c$) by passing to a new pair of axes parallel to the first with

$$\frac{V^2 \sin\alpha \cos\alpha}{g},\ \frac{V^2 \sin^2\alpha}{2g},$$

for the coördinates of the new origin, we have (Art. 100),

$$y + \frac{V^2 \sin^2\alpha}{2g} = \tan\alpha\left(x + \frac{V^2 \sin\alpha\cos\alpha}{g}\right)$$
$$- \frac{g}{2V^2\cos^2\alpha}\left(x + \frac{V^2\sin\alpha\cos\alpha}{g}\right)^2.$$

After reduction this becomes

$$x^2 = -\frac{2V^2\cos^2\alpha}{g}y,$$

which is seen to be a parabola convex upward with its vertex at the origin of coördinates. (Art. 129.)

This curve is approximately shown in a stream of water issuing from a hose. It may also be traced by watching a tennis-ball or base-ball as the ball moves through the air.

**68.** To find the **horizontal range,** we put $y = 0$ in equation ($c$); then $x = \frac{V^2 \sin 2\alpha}{g}$. This value is greatest when $\sin 2\alpha$ is greatest; that is, when $\alpha$, the angle of projection, is 45°.

It may be noticed that persons skilled in throwing have learned from experience that in order to throw as far as possible the ball or stone must be thrown in a direction about half way between horizontal and "straight up."

**69.** To find the range on an inclined plane let the straight line $y = x \tan\beta$ express the slope of the plane. We have then to find where the line $y = x\tan\beta$ cuts the parabola

$$y = x\tan\alpha - \frac{g}{2V^2\cos^2\alpha}x^2.$$

Eliminating $y$, we obtain

$$x = \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos\beta},$$

the abscissa of the point of intersection. The distance from the point of projection to this point of intersection is therefore

$$x\sec\beta, \text{ which equals } \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

To find the particular value of $\alpha$ that will make this distance a maximum, we must view this expression as a function of $\alpha$ and equate the first derivative to zero; that is, if $R$ is the range, $\frac{dR}{d\alpha} = 0$ is the condition for a maximum (or minimum). (Art. 41.)

We have then

$$\frac{dR}{d\alpha} = \frac{d}{d\alpha}x\sec\beta$$

$$= \frac{-2V^2\sin\alpha\sin(\alpha-\beta) + 2V^2\cos\alpha\cos(\alpha-\beta)}{g\cos^2\beta}.$$

Equating this to zero and reducing,

$$\cos\alpha\cos(\alpha-\beta) - \sin\alpha\sin(\alpha-\beta) = 0;$$

that is, $$\cos[\alpha + (\alpha-\beta)] = 0,$$

and hence $$2\alpha - \beta = 90°,$$

$$\alpha = \frac{90° + \beta}{2} = \beta + \tfrac{1}{2}(90° - \beta).$$

Therefore the direction of projection which secures the greatest range on a given inclined plane bisects the angle between the vertical and the inclined plane.

The student should of course raise the question: How do we know that the above equation of condition gives a value of $\alpha$ that secures a maximum range instead of a minimum?

**70.** Suppose that a piece of smooth wire or small-bore tubing, smooth on the inside, is bent into the shape of some plane curve and hung up vertically. Further, suppose that a bead is strung on the wire, or a small ball dropped into the tube. The body, say the ball, will slide downward under the action of gravity, but it will be obliged to follow a certain path. What will its velocity be at any point $P$?

Draw the usual axes in the vertical plane in which the curve lies. Let $A$ be the position of the body when $t = 0$; $P$ its position $(x, y)$ at any time $t$; and let arc $AP = s$. If $\alpha$ is the angle which the tangent at the point $P$ makes with the $x$-axis, $g \sin \alpha$ is the acceleration along the curve at $P$. But $\sin \alpha = -\frac{dy}{ds}$; hence

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = -g\frac{dy}{ds}.$$

Multiplying by $2\,ds$ and integrating, we obtain

$$\left(\frac{ds}{dt}\right)^2 = v^2 = -2gy + C.$$

If we call the ordinate of the point $A$ $y_0$ and the velocity at $A$ $v_0$, we have

$$v_0^2 = -2gy_0 + C;$$

and therefore $$v^2 - v_0^2 = 2g(y_0 - y).$$

Now let the ordinate $y_0$ be produced upward to a point $B$, making $AB = h$, the height from which the body falling freely would have to fall in order to acquire the velocity $v_0$. Draw $BN$ a line parallel to the $x$-axis. Let $C$ be the point where the ordinate $y$ produced meets the line $BN$. $v_0^2 = 2gh$ (Art. 60 (c)). Substituting this value of $v_0^2$ in the equation above,

$$v^2 = 2gh - 2g(y - y_0) = 2g(h + y_0 - y) = 2g \cdot PC.$$

Hence the velocity at any point $P$ is the same as the velocity that would have been acquired had the body fallen directly from the line $BN$ to $P$.

**71.** Let us now limit the case to motion in a vertical circle. Instead of having the ball slide in a circular tube we can just as well secure circular motion by attaching the ball to the end of a string whose other end is fastened at the center of the circle. We now have a **pendulum**. Let $C$ be the center of the circle; $O$ its lowest point; $OX$ the $x$-axis, and $OY$ the $y$-axis.

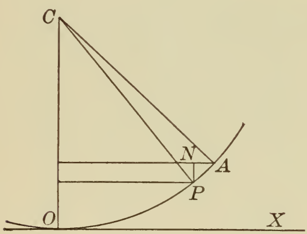


Fig. 7.

Let $A$ be the starting point of the ball; then at $A$ $t=0$ and $v_0 = 0$. Let $P$ be its position and $v$ its velocity at any time $t$. Also, let $\theta =$ angle $PCO$ and $\alpha =$ angle $ACO$; $s =$ arc $AP$ and $l = PC$, the length of the string.

By the preceding article,

$$\left(\frac{ds}{dt}\right)^2 = v^2 = 2g \cdot PN = 2gl(\cos\theta - \cos\alpha);$$

but
$$\left(\frac{ds}{dt}\right)^2 = l^2\left(\frac{d\theta}{dt}\right)^2;$$

hence
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\alpha)$$
$$= \frac{4g}{l}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right),$$

and therefore
$$\frac{d\theta}{dt} = -2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}.$$

Notice that after extracting the square root only the minus sign is used, because $dt$ is positive and $d\theta$ is negative.

We now have
$$\sqrt{\frac{g}{l}}\,dt = -\frac{d\theta}{2\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}};$$

that is,
$$\sqrt{\frac{g}{l}}\,t = -\int_\alpha^\theta \frac{d\theta}{2\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}.$$

The expression here presented for integration looks quite simple, but it cannot be expressed in finite terms by means of the ordinary algebraic or trigonometric functions. If, however, we expand $\sin^2\frac{\alpha}{2}$ by Maclaurin's theorem (Art. 34, ex. 14), and then take $\alpha$ so small that we may neglect powers of $\alpha$ (and $\theta$) beyond the second, we shall have
$$4\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right) = \alpha^2 - \theta^2.$$

The above integral then becomes

$$\sqrt{\frac{g}{l}}\,t = -\int_{\alpha}^{\theta} \frac{d\theta}{\sqrt{\alpha^2 - \theta^2}},$$

and this is integrated by formula $11_1$, Chap. V, so that we obtain

$$\sqrt{\frac{g}{l}}\,t = \cos^{-1}\frac{\theta}{\alpha}\Big]_{\alpha}^{\theta} = \cos^{-1}\frac{\theta}{\alpha} - \cos^{-1}\frac{\alpha}{\alpha} = \cos^{-1}\frac{\theta}{\alpha}.$$

Solving for $\theta$, $\qquad \theta = \alpha \cos\sqrt{\frac{g}{l}}\,t.$

When $\theta = 0$, $\sqrt{\frac{g}{l}}\,t = \cos^{-1} 0 = \frac{\pi}{2}$;

hence $t_0$, the time from $A$ to $O$, is $\frac{\pi}{2}\sqrt{\frac{l}{g}}$.

If $T$ be the time of an oscillation from $A$ to $A'$ (on the other side of $O$),

$$T = \pi\sqrt{\frac{l}{g}}.$$

This result is true only when $\alpha$ is small, as above shown. It is independent of $\alpha$, and therefore the time of an oscillation is the same for all small arcs in the same circle. That is, if $\alpha$ and $\alpha'$ are two small but unequal arcs, the times of oscillation for the same pendulum are equal.

**72.** It will be noticed that the equation

$$\theta = \alpha \cos\sqrt{\frac{g}{l}}\,t$$

is of the form of the equation expressing simple harmonic motion; therefore the pendulum-bob has simple harmonic motion in a circle which lies in a plane passing through $OX$ perpendicular to $OY$. The radius of this circle is $\alpha$, and the displacement at any time $t$ is $\theta$. If $\alpha$ is given in degrees, it must be divided by the radian ($57°.295779\cdots$). For instance, if $l = 50$ inches and $\alpha = 1°$, the radius of the circle across which the harmonic motion takes place is $\frac{50\,(1°)}{57.°+} = \frac{50}{57}$ inches, approximately.

**73. Areas.** Let $PS$ be a portion of the curve $y = f(x)$; and let it be required to find the area bounded by this arc, the ordinates $PM$ and $SN$, and the $x$-axis.

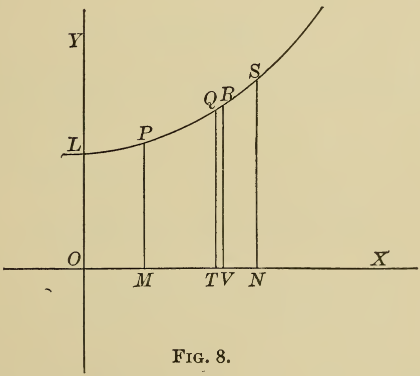


FIG. 8.

Let $OM = a$, $ON = b$, $OT = x$, and $OV = x + \delta x$; then $QT = y$, and $RV = y + \delta y$. If the area $OLQT$, any varying portion of the area $OLSN$, equals $A$, area $OLRV = A + \delta A$, and $\delta A = TQRV$. Now, if the short arc $QR$ were a straight line, the area $TQRS$ would be a trapezoid, and we should have

$$\delta A = \delta x \tfrac{1}{2}(QT + RV) = \delta x(y + \tfrac{1}{2}\delta y);$$

and $$\frac{\delta A}{\delta x} = y + \tfrac{1}{2}\delta y.$$

In the limit $QR$ becomes a straight line, and

$$\frac{dA}{dx} = y;$$

that is, $$dA = ydx = f(x)dx;$$

and this is a representative strip taken anywhere in the area $OLSN$.

Suppose $$\int f(x)dx = \phi(x) + C;$$

then $$A = \phi(x) + C.$$

Since we are measuring areas from the $y$-axis,

when $$x = 0,\ A = 0;$$

when $$x = a,\ A = \text{area } OLPM;$$

when $$x = b,\ A = \text{area } OLSN;$$

therefore $$\text{area } OLSN = \phi(b) + C,$$

$$\text{area } OLPM = \phi(a) + C.$$

Subtracting this last expression from the one preceding it,

$$\text{area } OLSN - \text{area } OLPM = \text{area } MPSN$$

$$= \phi(b) - \phi(a) = \int_a^b f(x)dx.$$

We have, then, for the area between the ordinates, whose distances from the $y$-axis are $a$ and $b$ respectively, the definite integral

$$\int_a^b f(x)dx.$$

**74.** For example, suppose we wish to find the area bounded by the parabola $y^2 = 4px$, the $x$-axis, and any ordinate $y'$ (the accompanying abscissa being $x'$).

$$A = \int_0^{x'} ydx = \int_0^{x'} 2p^{\frac{1}{2}}x^{\frac{1}{2}}dx = \frac{4p^{\frac{1}{2}}x^{\frac{3}{2}}}{3}\Bigg]_0^{x'} = \tfrac{4}{3}p^{\frac{1}{2}}x'^{\frac{3}{2}}.$$

We notice that the rectangle $x'y' = 2p^{\frac{1}{2}}x'^{\frac{3}{2}}$; hence the area in question equals two-thirds the circumscribed rectangle.

**Examples.**

1. Find the area of the upper right-hand quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

In this case

$$A = \int_0^a ydx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi ab}{4}.$$

The area of the entire ellipse is therefore $\pi ab$. $\pi a^2$, the area of the circle $x^2 + y^2 = a^2$, may now be viewed as a special case of the ellipse in which $b = a$.

2. Find the area between the isotherm $pv = c$, the $v$-axis, and the two ordinates whose distances from the $p$-axis are $a$ and $b$ respectively. *Ans.* $c \log \frac{b}{a}$.

3. Find the area bounded by the $x$-axis and the curve $y = \sin x$, from $x = 0°$ to $x = 180°$.

*Ans.* 2.

**75. Mean values.** The mean or average value of $n$ quantities is the $n$th part of their sum. If the quantities to be averaged are successive values of a function of some variable, their magnitudes depend not only on the nature of the function, but also on the law of variation of the fundamental. Thus, suppose we have the isotherm $pv = c$ and wish to know the average pressure between the volumes $v_1$ and $v_2$. It is necessary to make some assumption in regard to the variation of $v$. (1) If its increments are supposed equal, we understand by the "mean value" of the pressure the average of the pressures corresponding to the arithmetic series: $v$, $v + dv$, $v + 2\,dv$, etc. (2) If the volume is assumed to depend on some other variable in such a manner that the abscissa increments are not equal, the mean value will now be the average of a new series of pressure ordinates corresponding to the new series of values of $v$ arising under the second assumption. Evidently the two means will, in general, be unequal; but one is just as properly the *average* as the other. An important illustration is afforded if we ask: what is the mean distance of a planet from the sun? If a planet moved in its elliptic orbit in such a way that the radius vector described equal angles in equal times, that is, if $\frac{d\theta}{dt}$, its angular velocity, were a constant, the mean length of its radius vector could be shown to be $a\sqrt{1-e^2}$, $a$ being the semi-major axis, and $e$ the eccentricity of its orbit. But we know that the law of gravitation requires that the *areal* velocity shall be a constant; that is, the radius vector describes equal areas, instead of equal angles, in equal times (Art. 88). In one case $\frac{d\theta}{dt}$ is constant; in the other, $\frac{1}{2}r^2\frac{d\theta}{dt}$ is constant. A little consideration will show that the mean value of $r$ cannot be the same in the two cases.

**76.** If $y = f(x)$ and all of the $dx$'s are equal, the average length of $y$ between $x = a$ and $x = b$ is at once found by dividing the area $\int_a^b f(x)dx$ by $b - a$; for returning to Fig. 8, if the area $MPSN$ be divided by its base $b - a$, the quotient is the altitude of an equivalent rectangle of base $b - a$; and the altitude of the rectangle is the average altitude of the strips represented by $TQRV$; that is, of the $y$'s.

**Examples.**

1. Find the average length of the ordinates of a semicircle, supposing the series taken equidistant. We have $x^2 + y^2 = r^2$; or, $y = \sqrt{r^2 - x^2}$; therefore

$$M = \frac{1}{2r}\int_{-r}^{r}\sqrt{r^2 - x^2}dx = \tfrac{1}{4}\pi r.$$

From this result it appears that the average ordinate equals the length of an arc of 45°.

2. Find the average length of the ordinates, supposing they are drawn through equidistant points on the circumference.

In this case

$$M = \frac{1}{\pi}\int_0^{\pi} r \sin\theta d\theta = \frac{2r}{\pi}.$$

3. Given $pv = c$; show that the mean pressure between the volumes $v_1$ and $v_2$ is $\frac{c}{v_2 - v_1}\log\frac{v_2}{v_1}$, $v$ changing by equal increments.

4. A particle has simple harmonic motion. Find its mean velocity as it passes from the extremity of the radius to the center of the circle.

**77.** The above geometric conception of mean values may be adopted when a function is expressed in polar coördinates.

If $r = f(\theta)$, let $x$ be written for $\theta$, and $y$ for $r$, so that we have $y = f(x)$. This equation furnishes a curve which sustains peculiar relations to the original polar curve. The radii vectores lose their fan-shaped arrangement, and are placed parallel and equidistant (if $\theta$ is an equicrescent variable) with their extremities on a common line, the $x$-axis. The pole may be viewed as developing into this axis, — just as if a draw-string were let out, — while a circle of unit radius with the pole as center develops into a straight line parallel to the $x$-axis, the radii vectores keeping their position of perpendicularity with respect to the circumference of the circle. The mean value of the radius vector then becomes $\frac{1}{b-a}\int_a^b f(x)\,dx$, as before.

For example, to find $r_0$, the mean length of the radius vector of the ellipse $r = \frac{a(1-e^2)}{1+e\cos\theta}$, $\theta$ being an equicrescent variable, we have, using one-half of the ellipse,

$$r_0 = \frac{1}{\pi}\int_0^\pi \frac{a(1-e^2)}{1+e\cos x}dx = a\sqrt{1-e^2}.$$

The radii vectores, now in the *rôle* of ordinates, are distributed at equal intervals through an area $A$ whose base is $\pi$.

*Example.* Find the average length of the radius vector of the cardioid $r = a(1-\cos\theta)$.

$$M = \frac{1}{\pi}\int_0^\pi a(1-\cos x)\,dx = \frac{a}{\pi}\Big[x - \sin x\Big]_0^\pi = a.$$

**78. Work.** If a force $F$ acts on a body of mass $m$, giving it an acceleration $\frac{d^2s}{dt^2}$,

$$F = m\frac{d^2s}{dt^2}.$$

Multiplying by $ds$,

$$Fds = m\frac{d}{dt}\left(\frac{ds}{dt}\right)ds.$$

Integrating between the limits $v$ and $V$, $V$ being the velocity when $s=0$, and $v$ the velocity when $s=s$,

$$\int Fds = \tfrac{1}{2}m(v^2 - V^2).$$

If $$V = 0,\ \int Fds = \tfrac{1}{2}mv^2.$$

$Fds$ is defined as the **work** done on the body $m$ as it is moved through the space $ds$. $\int Fds$ is the work done in moving the body over the arc $s$.

$\frac{1}{2}mv^2$ is defined as the **kinetic energy** which the body possesses because work has been expended upon it, the kinetic energy representing the work stored up in the body. In order to perform the operation indicated by $\int Fds$ we of course need to know what function $F$ is of $s$ in case $F$ is a variable depending on $s$. Suppose that $F = \phi(s)$ and $W$ represents the work; then

$$W = \int_{s_1}^{s_2} \phi(s) ds.$$

From this it appears that work can be represented by an *area* referred to a *space* axis ($x$-axis), and a *force* axis ($y$-axis).

**79.** In fact, the integral for *area* is seen to be representative of all definite single integrals, these integrals taking the general form $\int_{x'}^{x''} f(x) dx$. The primary or horizontal axis is named for the quantity which $x$ denotes, and the secondary or vertical axis is named for the function of $x$. The integral itself is then represented by the area $MPSN$. (Fig. 8.)

**80. Lengths of curves.** Referring to Fig. 2, Art. 38, it is seen that

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy;$$

hence, if $s$ is the length of an arc from the point $(x', y')$ to the point $(x'', y'')$,

$$s = \int_{x'}^{x''} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{y'}^{y''} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Convenience must decide which of the formulas we shall use in any given example.

**Examples.**

1. Find the length of the catenary $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$, the curve in which a uniform chain hangs.

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right);$$

therefore $$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

and $$s = \int_0^x \frac{1}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) dx = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

2. Find the circumference of the circle $x^2 + y^2 = r^2$.

We have $\frac{dy}{dx} = -\frac{x}{y}$; then, if $AB$ is the first quadrantal arc of the circle,

$$\text{arc } AB = \int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^r \sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= \int_0^r \sqrt{\frac{x^2 + y^2}{y^2}}\, dx = r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}}$$

$$= r\left[\sin^{-1}\frac{x}{r}\right]_0^r = \frac{\pi r}{2}.$$

Therefore the circumference of the circle, $4\,AB$, $= 2\,\pi r$.

**81. Volumes of revolution; areas of surfaces of revolution.**

If a plane curve revolves around any line in its plane as an axis, it is evident that the figure generated is such that any cross-section of it by a plane at right angles to the axis is a circle. The *volume* may be found by taking the axis of revolution as the $x$-axis and adding together layers $dx$ in thickness. The area of any cross-section is $\pi y^2$. If $V$ represents volume, we have then

$$V = \int_a^b \pi y^2 dx,$$

in which the equation to the generating curve is

$$y = f(x).$$

Similarly, the *surface* may be found by noticing that the arc $\delta s$ generates the frustum of a cone whose surface is known from elementary geometry to be

$$\pi \left[ y + (y + \delta y) \right] \delta s;$$

or, in the limit, $2\pi y ds$; so that, if $S$ is the area of the surface,

$$S = \int_{x'}^{x''} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

**Examples.**

1. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ revolves about its major axis. What is the volume generated?

$$\tfrac{1}{2} V = \int_0^a \pi y^2 dx = \int_0^a \pi \frac{b^2}{a^2}(a^2 - x^2)\, dx = \frac{\pi b^2}{a^2}\int_0^a (a^2 - x^2)\, dx$$

$$= \frac{\pi b^2}{a^2}\left[ a^2 x - \frac{x^3}{3} \right]_0^a = \frac{2\pi a b^2}{3}.$$

The entire volume is therefore $\frac{4}{3}\pi ab^2$.

The volume of the sphere, $\frac{4}{3}\pi a^3$, is a special case, in which $b = a$.

2. Find the area of the surface generated as the ellipse revolves about its major axis.

$$\begin{aligned}\tfrac{1}{2}S &= \int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \frac{2\pi b}{a^2}\int_0^a \left[a^4-(a^2-b^2)x^2\right]^{\frac{1}{2}}dx \\ &= \pi b\left[b+\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right];\end{aligned}$$

therefore the whole surface is

$$\begin{aligned}&2\pi b\left[b+\frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right] \\ &= 2\pi b\left[b+\frac{a^2}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b}{a}\right].\end{aligned}$$

3. Find the area of the surface of a sphere whose radius is $a$.

If we make $b = a$, we have, from the result in the preceding example, for the area of the surface of the sphere,

$$2\pi a\left[a+a^2\frac{\cos^{-1}\frac{a}{a}}{\sqrt{a^2-a^2}}\right] = 2\pi a\left[a+a^2\frac{0}{0}\right].$$

We must now find the value of

$$\frac{\cos^{-1}\frac{b}{a}}{\sqrt{a^2-b^2}} \text{ when } b = a.$$

Treating $b$ as a variable and applying the principle of Art. 32,

$$\left.\frac{\cos^{-}\frac{b}{a}}{\sqrt{a^2-b^2}}\right]_{b=a}=\left.\frac{\frac{d}{db}\left(\cos^{-1}\frac{b}{a}\right)}{\frac{d}{db}\sqrt{a^2-b^2}}\right]_{b=a}=\left.\frac{-\dfrac{\frac{1}{a}}{\sqrt{1-\frac{b^2}{a^2}}}}{-\dfrac{b}{\sqrt{a^2-b^2}}}=\frac{1}{b}\right]_a=\frac{1}{a}.$$

Therefore, $S=2\,\pi a\left(a+a^2\frac{1}{a}\right)=4\,\pi a^2.$

This result for the area of the surface of a sphere agrees, of course, with the one obtained by the method of elementary geometry.

**82.** The area integral $\int_a^b f(x)\,dx$ represents the sum of strips whose height is $y$ and breadth $dx$. We may reach the same result by starting with the elementary rectangle $dxdy$ and using two integral signs, — one to indicate that we add such rectangles together to make a strip $y$ in height, and a second to indicate that the strips are to be added together, making the area from $a$ to $b$ (Fig. 8). For example, the area of the ellipse may be found by adding together the areas $dxdy$ from the major axis to the curve itself; then adding together the strips from the minor axis to the end of the major axis. To indicate this double operation, we write

$$A=\int_0^a\int_0^y dxdy,$$

using the right-hand integral sign with $dy$.

Performing the first operation,

$$\int_0^a \int_0^y dxdy = \int_0^a ydx.$$

The remaining part of the work is the same as in Art. 74, example 2.

The above procedure in finding areas involves what is known as a **double integral.** Similarly, three successive indicated integrations constitute a **triple integral.** Examples of double integrals will occur in subsequent articles.

**83.*** Suppose a point to travel once round the closed oval area $A$, an indicator diagram, for instance, so as always to have the interior of the curve on the *left*

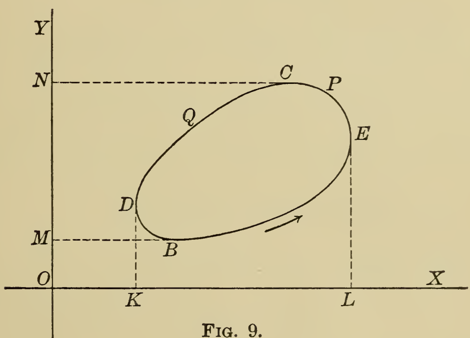


FIG. 9.

hand. Let $B$ be the minimum point, and $C$ the maximum point with respect to the $x$-axis; $D$ the minimum point, and $E$ the maximum point with respect to the $y$-axis.

* Greenhill's *Differential and Integral Calculus.*

Then $$A = \int\int dx dy = \int x dy,$$

taken round the perimeter of the curve.

From $B$ to $C$ along $BPC$, $dy$ is positive, and

$$\int x dy = \text{area } MBPCN.$$

From $C$ to $B$ along $CQB$, $dy$ is negative, and

$$\int x dy = -\text{ area } MBQCN;$$

so that, taken round the curve,

$$\int x dy = \text{area } MBPCN - \text{area } MBQCN = A,$$

the area of the closed curve.

But $$\int\int dx dy = \int y dx;$$

and from $E$ to $D$ along $EPD$, $dx$ is negative, so that

$$\int y dx = -\text{ area } LEQDK;$$

and from $D$ to $E$ along $DBE$, $dx$ is positive, so that

$$\int y dx = \text{area } LEBDK;$$

and therefore, taken round the curve,

$$\int y dx = -A.$$

Therefore taken round the curve,

$$\int (y dx + x dy) = 0;$$

and $y dx + x dy = d(xy)$ is called a **perfect differential.** Its integral between two limits is independent of the intermediate values of $x$ and $y$ and of the path described between the limits; so that, taken round any closed path, the integral is zero.

When Fig. 9 represents an indicator diagram, and $KL$ the reduced stroke of the piston, while the ordinate $y$ represents the pressure of the steam, the pencil will describe the contour with the area to the left, when the steam pressure is urging the piston from $L$ to $K$. The diagram taken on the return stroke from the other end of the cylinder will be described in the opposite sense, with the area on the right hand of the describing pencil.

**84. Moment of inertia.** When a rigid body rotates about an axis, the linear velocity of any particle of the body is

$$v = \frac{ds}{dt} = \frac{r d\theta}{dt} = r\omega,$$

$\omega$ being the angular velocity of the particle, and $r$ its distance from the axis. Its *kinetic energy of rotation* is therefore $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$, $m$ being the mass of the particle; and the kinetic energy of rotation of the whole body is

$$\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 + \frac{1}{2}m''v''^2 + \cdots$$
$$= \Sigma\frac{1}{2}mv^2 = \Sigma\frac{1}{2}m\omega^2r^2 = \frac{1}{2}\omega^2\Sigma mr^2;$$

that is, one-half the product of the square of the angular velocity and $\Sigma mr^2$.

The symbol $\Sigma$ is used to indicate a polynomial in which the terms are similarly constituted, as in the case before us. Since such an expression as $\Sigma\frac{1}{2}mv^2$ is in reality a polynomial, only common factors can be removed and placed before $\Sigma$, the symbol of summation. Thus in $\Sigma\frac{1}{2}m\omega^2r^2$, $\frac{1}{2}$ is, of course, a common factor; $\omega^2$ is a common factor because the rotating body is supposed to be rigid, and consequently all of its parts have the same angular velocity; but $m$ is not a common factor because it is not supposed that all of the particles have equal masses; neither is $r$ a common factor, for the particles are at different distances from the axis of rotation.

The quantity $\Sigma mr^2$ is called the **moment of inertia** of the body with respect to the axis, and is seen to be the sum of the products obtained by multiplying the mass of each particle by the square of its distance from the axis.

If a body rotates with a given angular velocity about different axes, the kinetic energy of rotation with respect to any axis must be proportional to $\Sigma mr^2$; consequently, the moment of inertia measures the capacity of a body to store up kinetic energy during rotation about the axis with respect to which the moment of inertia is taken.

### Examples.

**85.** 1. A sheet of metal, rectangular in shape and of uniform density, is made to rotate about an axis coinciding with one end. What is its moment of inertia?

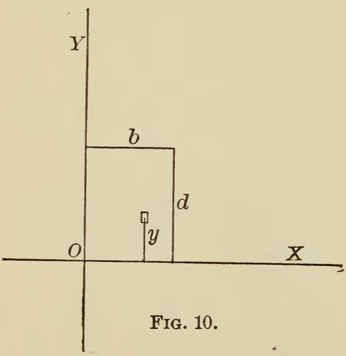

Fig. 10.

Take the axis of rotation for the $x$-axis with the origin at the left-hand corner of the rectangle. Let $b$ be the breadth and $d$ the height of the rectangle. If $\rho$ is the density of the metal, $\rho dy dx$ is the mass of the indefinitely small rectangle $dy\, dx$ cut anywhere from the sheet; and $(\rho dydx)y^2$ is the moment of inertia of this small piece. Hence the moment of inertia of the entire sheet becomes

$$\int_0^d\int_0^b \rho y^2 dydx = \rho\int_0^d\int_0^b y^2 dydx$$

$$= \rho b\int_0^d y^2 dy = \frac{\rho b d^3}{3}.$$

From this example it is plain that in all cases in which the density is constant throughout the body, the density factor may as well be set aside until the integration is completed. If, however, the density varies from point to point, so that $\rho$ is some specified function of $x$ and $y$, it must be kept under the sign of integration and be taken account of in the process of integrating.

2. A straight slender rod of length $l$, whose density varies directly as the distance from one end, rotates about an axis perpendicular to it and passing through the end having the least density. What is the moment of inertia with reference to this axis?

Take the given axis as the $x$-axis, with the origin at the end of the rod.

$\rho \propto y$; therefore $\rho = ky$ if $k$ is the density at a unit's distance from the end. Then the moment of inertia is

$$\int_0^l \rho y^2 dy = \int_0^l k y^3 dy = \frac{kl^4}{4}.$$

3. Find the moment of inertia of a circle with reference to an axis through its center and perpendicular to it, $\rho$ being a constant.

Let $R$ be the radius of the circle, $r$ the distance of any particle from the axis, and $\theta$ the variable angle measured from some chosen radius. Consider an elementary portion bounded by the circles whose radii are $r$ and $r + dr$, and by the radii forming the angle $d\theta$. In the limit this bit of area becomes the rectangle $(rd\theta)dr$; hence, the integral is

$$\int_0^R \int_0^{2\pi} r^2(rdrd\theta) = 2\pi \int_0^R r^3 dr = \frac{\pi}{2} R^4;$$

and therefore the moment of inertia is $\frac{\pi \rho R^4}{2}$.

**86. Kepler's laws.** It is shown in works on the determination of orbits* that the equations for the undisturbed motion of a planet or comet relative to the sun are:

$$\frac{d^2x}{dt^2} + k^2(1 + m)\frac{x}{r^3} = 0, \qquad (1)$$

$$\frac{d^2y}{dt^2} + k^2(1 + m)\frac{y}{r^3} = 0, \qquad (2)$$

$$\frac{d^2z}{dt^2} + k^2(1 + m)\frac{z}{r^3} = 0, \qquad (3)$$

in which $x$, $y$, $z$ are the coördinates of the heavenly body referred to the sun as origin, $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are the accelerations parallel to the three axes of reference, $r$ is the distance of the body from the sun, $k^2$ is the mass of the sun, and $m$ the ratio of the mass of the body to the mass of the sun. Having these three equations, we

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* Watson, *Theoretical Astronomy;* Dziobek, *Planeten-Bewegungen;* Tisserand, *Détermination des Orbites.*

can at once establish Kepler's laws. Arts. 87, 88, 89, 90, 93 are taken, with slight changes, from Watson's *Theoretical Astronomy*.

**87.** If we multiply equation (1) of the preceding article by $y$, and equation (2) by $x$, and subtract the last product from the first, we shall have, after integrating the result,

$$\frac{xdy - ydx}{dt} = C,$$

$C$ being the constant of integration.

In a similar manner we obtain

$$\frac{xdz - zdx}{dt} = C', \quad \frac{ydz - zdy}{dt} = C''.$$

If we multiply these three equations respectively by $z$, $-y$, and $x$, and add the products,

$$Cz - C'y + C''x = 0.$$

This is the equation to a plane passing through the origin of coördinates (Art. 142). Since $x$, $y$, $z$ are the coördinates of the heavenly body, it must remain in this plane. The path of the heavenly body relative to the sun is therefore a plane curve, and *the plane of the orbit passes through the center of the sun.*

**88.** If we multiply equations (1), (2), and (3) respectively by $2\,dx$, $2\,dy$, and $2\,dz$, take the sum and integrate, we have

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} + 2\,k^2(1+m)\int\frac{xdx + ydy + zdz}{r^3} = 0. \quad (4)$$

But $$r^2 = x^2 + y^2 + z^2,$$

therefore $$rdr = xdx + ydy + zdz.$$

Introducing this value of $xdx + ydy + zdz$ into equation (4) and performing the integration indicated, we have

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2k^2(1+m)}{r} + h = 0, \tag{5}$$

$h$ being the constant of integration.

If we add together the squares of the expressions for $C$, $C'$, $C''$, and put $C^2 + C'^2 + C''^2 = 4f^2$, we shall have

$$\frac{(x^2+y^2+z^2)(dx^2+dy^2+dz^2)}{dt^2} - \frac{(xdx+ydy+zdz)^2}{dt^2} = 4f^2;$$

that is, $$r^2\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{r^2dr^2}{dt^2} = 4f^2. \tag{6}$$

If we now represent by $dv$ the infinitely small angle contained between two consecutive radii-vectores $r$ and $r + dr$, since $dx^2 + dy^2 + dz^2$ is the square of $ds$, the element of path described by the body, we shall have

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2dv^2.$$

Substituting this value of $dx^2 + dy^2 + dz^2$ in equation (6),

$$r^2dv = 2fdt. \tag{7}$$

The quantity $r^2dv$ is double the area included by the element of path described in the element of time $dt$, and by the radii-vectores $r$ and $r + dr$. See Fig. 4, Art. 46. The area of the triangle $POP' =$ area $POQ$ + area $PQP'$; but in the limit area $PQP'$ vanishes, and area $POQ = \frac{1}{2} OP(PQ) = \frac{1}{2} r(rd\theta)$. Writing equation (7) in the form

$$\frac{\frac{1}{2} r^2 dv}{dt} = f,$$

the quantity $\dfrac{\frac{1}{2} r^2 dv}{dt}$ is the area described by the radius-vector in the time $dt$, divided by the time, and is defined as the **areal velocity**. Since $f$ is a constant, we conclude that *the radius-vector of a planet or comet describes equal areas in equal intervals of time.* (Kepler's second law.)

**89.** Combining equations (5) and (6) so as to eliminate $\dfrac{dx^2 + dy^2 + dz^2}{dt^2}$, and solving for $dt$, we have

$$dt = \frac{rdr}{\sqrt{2rk^2(1+m) - hr^2 - 4f^2}}. \qquad (8)$$

Substituting this value of $dt$ in equation (7),

$$\frac{dr}{dv} = \frac{r\sqrt{2rk^2(1+m) - hr^2 - 4f^2}}{2f}. \qquad (9)$$

We have seen (Art. 46) that the condition that $r$ shall be a maximum or minimum is $\dfrac{dr}{rd\theta} = 0$. With the notation of the present article, $v = \theta +$ some constant; therefore $dv = d\theta$, and we have, in order to find the maximum and minimum values of $r$,

$$\frac{\sqrt{2rk^2(1+m) - hr^2 - 4f^2}}{2f} = 0;$$

that is, $\qquad 2rk^2(1+m) - hr^2 - 4f^2 = 0.$

If $r_1$ and $r_2$ represent the two roots of this quadratic equation,

$$r_1 = \frac{k^2(1+m)}{h} + \sqrt{-\frac{4f^2}{h} + \frac{k^4(1+m)^2}{h^2}},$$

$$r_2 = \frac{k^2(1+m)}{h} - \sqrt{-\frac{4f^2}{h} + \frac{k^4(1+m)^2}{h^2}}.$$

Since the equation of condition yields only two values of $r$, the orbit cannot have more than two apsidal points. If it is a closed curve and not a circle, it must evidently have two, rather than one, such points. The point corresponding to $r_1$, the maximum value of $r$, is called the **aphelion**, and the point corresponding to $r_2$ is the **perihelion**.

**90.** If we put

$$\frac{k^2(1+m)}{h} + \sqrt{-\frac{4f^2}{h} + \frac{k^4(1+m)^2}{h^2}} = a(1+e)$$

and $$\frac{k^2(1+m)}{h} - \sqrt{-\frac{4f^2}{h} + \frac{k^4(1+m)^2}{h^2}} = a(1-e)$$

and add the two expressions, we have

$$h = \frac{k^2(1+m)}{a}.$$

Also, taking the difference of the two expressions and substituting the value of $h$ just found,

$$4f^2 = ak^2(1+m)(1-e^2) = k^2p(1+m)$$

if $p$ be written for $a(1-e^2)$.

Substituting these values of $h$ and $4f^2$ in equation (9) it becomes

$$dv = \frac{\sqrt{p}\,dr}{r\sqrt{2r - \frac{1}{a}r^2 - p}} = -\frac{\frac{p}{e}\,d\,\frac{1}{r}}{\sqrt{1 - \left(\frac{p}{e}\frac{1}{r} - \frac{1}{e}\right)^2}},$$

the integral of which gives

$$v = \cos^{-1}\frac{1}{e}\left(\frac{p}{r} - 1\right) + \omega,$$

$\omega$ being the constant of integration; and, therefore,

we have $$\frac{1}{e}\left(\frac{p}{r} - 1\right) = \cos\,(v - \omega);$$

that is, solving for $r$,

$$r = \frac{p}{1 + e\cos(v - \omega)}. \qquad (10)$$

This expression is seen to be the polar equation to a conic section (Art. 115), the pole being at the focus, $p$ being the semi-latus rectum, $e$ the eccentricity, and $\omega$ the angle at the focus between the major axis and a fixed line in the plane of the orbit. $v$, the vectorial angle, is measured from this latter line.

If $\omega = 0$, equation (10) becomes

$$r = \frac{p}{1 + e\cos v} = \frac{a(1 - e^2)}{1 + e\cos v}. \qquad (11)$$

In this case $v$ is called the **true anomaly**. We now conclude that *the orbit of a heavenly body revolving around the sun is a conic section with the sun in one of the foci.* (Kepler's first law.)

**91.** The planets revolve around the sun in ellipses, and these ellipses are, as a rule, characterized by small eccentricities. Thus the eccentricity of the earth's orbit is at present 0.01677. Of all the major planets Mercury has the most elliptic orbit, its eccentricity being 0.2056.

The orbits of comets, on the other hand, may be described as parabolic, by which we mean that they are either ellipses of great eccentricity (almost unity), or hyperbolas whose eccentricity differs but little from unity. In many cases the eccentricity cannot be found to differ from unity; the orbit is then of course described as a parabola. Of the periodic comets which have been observed at more than one perihelion passage, Tempel's comet has the least eccentricity, namely: 0.4051.

**92.** In putting $r_1$ and $r_2$ equal to $a(1+e)$ and $a(1-e)$ respectively, the argument has much the air of begging the question, seeming to assume that the orbit is a conic section and then using the assumption in the proof. But it is to be noted that when we adopt the expressions $a(1+e)$ and $a(1-e)$, $e$ does not mean eccentricity, neither does $a$ mean semi-major axis. They do not bear these meanings until in equation (10) we identify them with the constants in the polar equation of coördinate geometry.

**93.** If the values of $h$ and $4f^2$, as found above, are introduced into equation (8), we have

$$dt = \frac{\sqrt{a}}{k\sqrt{1+m}} \cdot \frac{rdr}{\sqrt{a^2e^2-(a-r)^2}},$$

which may be written

$$dt = -\frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \cdot \frac{\left(1 - \frac{a-r}{a}\right) d\left(\frac{a-r}{ae}\right)}{\sqrt{1 - \left(\frac{a-r}{ae}\right)^2}},$$

or

$$dt = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \left[ \frac{-d\left(\frac{a-r}{ae}\right)}{\sqrt{1 - \left(\frac{a-r}{ae}\right)^2}} + e \frac{\frac{a-r}{ae} d\left(\frac{a-r}{ae}\right)}{\sqrt{1 - \left(\frac{a-r}{ae}\right)^2}} \right];$$

the integration of which gives

$$t = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \left[ \cos^{-1}\left(\frac{a-r}{ae}\right) - e\sqrt{1 - \left(\frac{a-r}{ae}\right)^2} \right] + C. \quad (12)$$

When the heavenly body is in perihelion, $r = a(1-e)$, and the integral reduces to $t' = C$; therefore, if we denote the time from perihelion by $t_0$, we have

$$t_0 = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \left[ \cos^{-1}\left(\frac{a-r}{ae}\right) - e\sqrt{1 - \left(\frac{a-r}{ae}\right)^2} \right]. \quad (13)$$

We here integrate between the limits $t'$ and $t$, with $t - t' = t_0$.

In aphelion $r = a(1+e)$; putting this value of $r$ into equation (13), and denoting by $\frac{1}{2}\tau$ the time from perihelion to aphelion, we have

$$\tfrac{1}{2}\tau = \frac{a^{\frac{3}{2}}}{k\sqrt{1+m}} \pi. \quad (14)$$

According to Kepler's second law, the time from aphelion to perihelion must equal the time from perihelion to aphelion; therefore $\tau$ is the time of a complete revolution.

The time of a complete revolution is termed the **periodic time.**

From equation (14)

$$\tau^2 = 4\pi^2 \frac{a^3}{k^2(1+m)}; \tag{15}$$

and for a second planet,

$$\tau'^2 = 4\pi^2 \frac{a'^3}{k^2(1+m')}. \tag{16}$$

Comparing equations (15) and (16), we see that

$$\frac{(1+m)\tau^2}{(1+m')\tau'^2} = \frac{a^3}{a'^3}. \tag{17}$$

If the masses of the two planets are very nearly the same, we may take $1 + m = 1 + m'$; and hence, in this case, it follows that *the squares of the periodic times of two planets are to each other as the cubes of the semi-major axes.* (Kepler's third law.)

# CHAPTER IV.

## ANALYTIC GEOMETRY.

**94.** In this chapter it is proposed to present the elementary principles of analytic (coördinate) geometry, with especial reference to conic sections.

The Cartesian system* of coördinates has already been explained in Art. 36. Here, as there, we shall speak of the curve $F(x, y) = 0$, the curve $y = f(x)$, the line $ax + by + c = 0$, etc., instead of saying "the curve which the equation $F(x, y) = 0$ represents," etc.

**95.** If the equations

$$y = f(x), \qquad (a)$$

$$y = \phi(x), \qquad (b)$$

are treated as simultaneous, the $x$ and $y$ of equation $(a)$ must mean the same as the $x$ and $y$ of equation $(b)$; consequently, as coördinates they are restricted to the point or points common to the two curves $(a)$ and $(b)$.

If equations $(a)$ and $(b)$ have been so combined as to eliminate one of the coördinates, say $y$, the $x$ of the resulting equation is the abscissa of the point of intersection of the two curves, and the curves intersect in as many real points as there are real roots of this new equation.

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* Called the "Cartesian system," after René Descartes (1596–1650), the inventor of coördinate geometry.

For example, if $y$ be eliminated between the two equations $x^2+y^2-4=0$ and $x+y-1=0$, we have $x^2-x-\frac{3}{2}=0$. $\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$ are therefore the abscissas of the points where $x^2+y^2-4=0$ and $x+y-1=0$ intersect.

**96.** If, however, equations ($a$) and ($b$) have been so combined that neither $x$ nor $y$ is eliminated, the $x$ and $y$ now refer primarily to the points common to the curves of ($a$) and ($b$); but we may treat them as a new $x$ and $y$, — the current coördinates of a point describing a new curve, which passes through the intersections of the curves ($a$) and ($b$).

For example, suppose we have the equations $y=2x-2$ and $2y=x+2$. Adding them, $y=x$, a straight line distinct from the given lines, but passing through their point of intersection.

**97.** If the coördinates of a given point satisfy a given equation, the point evidently lies on the curve which the equation represents. Conversely, if a point is on a curve, its coördinates will satisfy the equation to the curve.

If an equation $F(x, y)=0$ can be written

$$f(x, y)\phi(x, y)=0,$$

the curve of the given equation is made up of the combined curves of $f(x, y)=0$ and $\phi(x, y)=0$; for any point whose coördinates cause $f(x, y)$ to vanish, thus satisfying the equation $f(x, y)=0$, will also cause $f(x, y)\phi(x, y)$ to vanish. Hence, all points on $f(x, y)=0$ are also points on $f(x, y)\phi(x, y)=0$. Similarly, all points on $\phi(x, y)=0$ are points on $f(x, y)\phi(x, y)=0$.

Further, there are no other points on $f(x, y)\phi(x, y)=0$, because $f(x, y)\phi(x, y)$ cannot vanish except by the vanishing of either $f(x, y)$ or $\phi(x, y)$.

For example,

$$x^2+2y^2+3xy-x-y=(x+y)(x+2y-1);$$

hence, the curve which $x^2+2y^2+3xy-x-y=0$ represents is made up of the straight lines $x+y=0$ and $x+2y-1=0$.

(The term "curve" is here used as inclusive of straight lines.)

**98.** If we have the equations $y=f(x)$ and $y=\phi(x)$, the equation $y=f(x)\phi(x)$ represents a curve whose ordinate for any abscissa $x'$ is the product of the ordinates corresponding to $x'$ in the two primary curves.

For example, if $y=x$ and $y=\log x$, $y=x\log x$ is a third curve whose ordinate at any point equals the product of the corresponding ordinates.

In drawing such a set of curves to the same axes of reference it is well to use colored pencils or crayons. For instance, if the straight line $y=x$ is drawn in red, the logarithmic curve in yellow, and the curve $y=x\log x$ in blue, the resulting diagram appeals to the eye much more forcibly than if all were done in black or white.

**99.** If $y=f(x)$ and $y=\phi(x)$, the equation

$$y=f(x)+\phi(x)$$

represents a curve whose ordinate at any point is the sum of the corresponding ordinates of the given curves.

For example, the so-called "equation of time" is made up of two parts: one due to the eccentricity of the earth's orbit, the other to the obliquity of the ecliptic. If $E_1$ and $E_2$ represent these two parts respectively, $E$, the whole equation of time, equals $E_1 + E_2$. With a scale of dates one year long for the $x$-axis, and a scale marked to minutes for the $y$-axis, we may construct the curve of $E_1$ and also the curve of $E_2$. A third curve, whose ordinate for any date is the sum of the ordinates of the first two curves for that date, then represents $E$. See Young's *General Astronomy*, Fig. 64, edition of 1898.

The principles of this section and the preceding one can evidently be extended to such forms as $y = \frac{f(x)}{\phi(x)}$, $y = f(x) - \phi(x)$, etc.

**100.** If we move the $x$-axis parallel to itself through the distance $y'$, every ordinate is changed by the amount $y'$. Similarly, if the $y$-axis is moved parallel to itself through the distance $x'$, every abscissa is changed by the amount $x'$. So, if $X$ and $Y$ are the new current coördinates, $x = X + x'$ and $y = Y + y'$, in which $x'$ and $y'$ are the coördinates of the new origin referred to the old axes. Hence, if in any equation $F(x, y) = 0$, we write $x + x'$ for $x$, and $y + y'$ for $y$, so that the equation becomes $F(x + x', y + y') = 0$, the geometric result secured is a *change of origin* to a new point $(x', y')$, with new axes parallel to the old ones.

The new current coördinates may be written $x$, $y$, instead of $X$, $Y$, since they do not occur in connection with the old coördinates and therefore cannot be confused with them.

For example, $x^2 + y^2 = r^2$ being the equation to a circle with its center at the origin, $(x - a)^2 + (y - b)^2 = r^2$ is the same circle with its center at $(a, b)$. The coördinates of the *new* origin referred to the *old* axes are $-a, -b$.

**101.** Suppose the axes to rotate around the origin through the angle $\alpha$. The new coördinates of any point $P$ are

$$X = OB'; \quad Y = PB'.$$

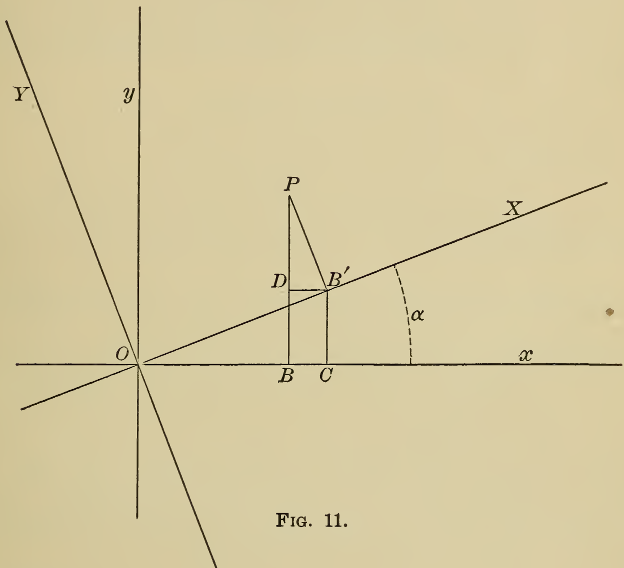


FIG. 11.

Now

$$OB' \cos\alpha = OC = OB + BC = x + BC = x + PB' \sin\alpha.$$

Therefore $\quad x = OB' \cos\alpha - PB' \sin\alpha$

that is, $\quad x = X\cos\alpha - Y\sin\alpha.$

Similarly, $\quad y = X\sin\alpha + Y\cos\alpha.$

Hence, if in any equation $F(x, y) = 0$ we write $x \cos \alpha - y \sin \alpha$ for $x$ and $x \sin a + y \cos \alpha$ for $y$, the geometric result is the rotation of the axes through the angle $\alpha$, in which $\alpha$ may have any value and be positive or negative.

**102.** In the formulas just derived, the old coördinates $x$ and $y$ are explicit functions of the new coördinates $X$ and $Y$. If we multiply the first formula by $\cos \alpha$ and the second by $\sin \alpha$, and add the products, we have

$$X = x \cos \alpha + y \sin \alpha.$$

Similarly, $$Y = -x \sin \alpha + y \cos \alpha.$$

The new coördinates are now explicit functions of the old ones.

**103.** The formulas derived in the three preceding articles are indispensable in astronomy. As an example of the use of the two in Art. 102, suppose a planet is referred to the line in which the plane of its orbit cuts the ecliptic as the $x$-axis, with a line at right angles to it in the ecliptic as the $y$-axis. The planet may be referred to a new $x$-axis having its positive end directed toward the vernal equinox, with a corresponding new $y$-axis, if we use the relations

$$X = x \cos ☊ - y \sin ☊,$$

$$Y = x \sin ☊ + y \cos ☊.$$

The axes are here moved backward, that is, in the negative direction, through the angle ☊ (the longitude of the ascending node), ☊ being the angle between the vernal end of the equinoctial line and the line passing through the sun and the point through which the planet moves in going from the south to the north side of the ecliptic.

**104.** We have seen (Art. 39) that $ax + by + c = 0$ represents a straight line because $\frac{dy}{dx} = -\frac{a}{b}$, a constant. It follows that if in any two equations

$$y = mx + n,$$

$$y = m'x + n',$$

$m' = m$, the lines are parallel, for they have the same slope.

Also, if $m' = -\frac{1}{m}$, the lines are at right angles to each other; for $m$ and $m'$ are now the tangents of $\alpha$ and $90° + \alpha$; since $\tan(90° + \alpha) = -\cot\alpha$.

**105.** If $x$ is put equal to zero in the equation $F(x, y) = 0$, the resulting value of $y$ must be the ordinate of the point where the curve crosses the $y$-axis. Now if $x = 0$ in $ax + by + c = 0$, $y = -\frac{c}{b}$; but $-\frac{c}{b}$ is the constant term when the equation is written in the form $y = -\frac{a}{b}x - \frac{c}{b}$. It follows that if any linear equation be written in the form $y = mx + n$, the constant term is the distance from the origin to the point where the line crosses the $y$-axis.

**106.** Suppose that $P'(x', y')$, $P''(x'', y'')$ are any two points on a straight line, while $P(x, y)$ is the moving point. From similar triangles (Fig. 12),

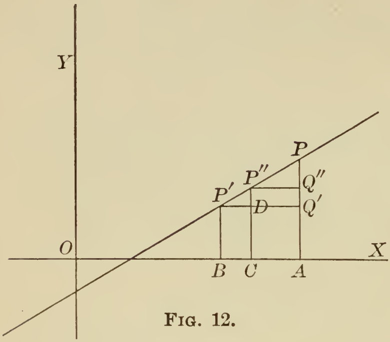


FIG. 12.

$$\frac{PQ'}{P'Q'} = \frac{P''D}{P'D};$$

that is,

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}$$

This equation we describe as the equation to the straight line in terms of the coördinates of two points through which it passes.

For example, the points $(-2, 1)$, $(3, -4)$ determine a line whose equation is

$$\frac{y - 1}{x - (-2)} = \frac{-4 - 1}{3 - (-2)},$$

which becomes, after reducing,

$$x + y + 1 = 0.$$

If $P'$ and $P''$ are indefinitely near each other, $y'' - y'$ becomes $dy$, and $x'' - x'$ becomes $dx$. Hence

$$y - y' = \frac{dy}{dx}(x - x'),$$

which is the equation to the straight line in terms of its gradient (or slope) and one point through which it passes.

**107.** Referring to Fig. 12, we see that if $D$ is the distance between any two points,

$$D=\sqrt{(x''-x')^2+(y''-y')^2}.$$

Also, if $P''$ is midway between $P$ and $P'$,

$$x''=\tfrac{1}{2}(x+x') \text{ and } y''=\tfrac{1}{2}(y+y');$$

that is, the coördinates of a point bisecting the distance between two points are the averages of the abscissas and of the ordinates, respectively, of the given points.

**108.** To find the perpendicular distance from a given point $(x', y')$ to a given line

$$y=mx+n, \qquad (1)$$

we write

$$y-y'=m(x-x'); \qquad (2)$$

a line passing through $(x', y')$ parallel to (1). The intercept of (1) on the $y$-axis is $n$, the intercept of (2) on the $y$-axis is $y'-mx'$, and the difference of these intercepts is $y'-mx'-n$. It is evident that if this difference of intercepts be multiplied by $\cos(\tan^{-1} m)$, that is, by $\dfrac{1}{\sqrt{1+m^2}}$, we shall have the perpendicular distance between the two lines, and hence the distance from $(x', y')$ to $y=mx+n$. The formula for the distance from any point $(x', y')$ to any line $y=mx+n$ is therefore

$$\frac{y'-mx'-n}{\sqrt{1+m^2}}.$$

Or, if the equation to the line be in the form

$$ax+by+c=0,$$

the formula is

$$\frac{y' + \frac{a}{b}x' + \frac{c}{b}}{\sqrt{1 + \left(\frac{a}{b}\right)^2}},$$

which equals

$$\frac{ax' + by' + c}{\sqrt{a^2 + b^2}}.$$

It appears, then, that we have simply to evaluate the function $ax + by + c$ for the coördinates of the given point and divide by the square root of the sum of the squares of the coefficients of $x$ and $y$.

### Exercises.

1. Given the equation $ax + by + c = 0$, show that if it be written in the form

$$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1,$$

the quantities standing beneath $x$ and $y$ are the intercepts on the axes of $x$ and $y$ respectively.

2. Write the formula for the distance from the origin to the line $ax + by + c = 0$.

3. Find the equation to a straight line which passes through a given point $(p, q)$ and makes equal angles with the axes.

4. Find the length of the perpendicular from the origin on the line $a(x - a) + b(y - b) = 0$. Also, find the portion of this line intercepted by the axes.

5. Write the equation to a line passing through the origin and making an angle of 120° with the $x$-axis.

6. The coördinates of the vertices of a triangle are $(1, 2)$, $(-3, \frac{1}{2})$, $(4, \sqrt{2})$; write the equations to its sides.

7. Find the equation to a line which passes through the intersection of the lines $x = a$, $x + y + a = 0$, and through the origin.

8. Show that the lines $y = 2x + 3$, $y = 3x + 4$, $y = 4x + 5$ all pass through one point.

9. Find the slope of the line $y = mx + 3$ in order that it may pass through the intersection of the lines $y = x + 1$ and $y = 2x + 2$.

10. Find the equation to the straight line which is equidistant from the two lines $y = mx + n \pm n'$.

11. If $\alpha$ is the angle between the lines $y = mx + n$ and $y = m'x + n'$, show that $\tan \alpha = \dfrac{m - m'}{1 + mm'}$.

**109. The ellipse.** Suppose that the circle $x^2 + y^2 = r^2$ is in a plane $M$ which makes an angle $\gamma$ with another plane $N$. Let the $x$-axis to which the circle is referred be parallel to plane $N$. If perpendiculars to $N$ are dropped from the extremities of the ordinates of the circle, the feet of these perpendiculars will form in $N$ a new curve which is the *projection* of the circle on the plane $N$; and any ordinate $y'$ of the circle becomes the ordinate $y' \cos \gamma$ in the new curve. Hence, if we write the circle-equation in the form $y = \pm\sqrt{a^2 - x^2}$, the new curve, called the **ellipse**, has for its equation

$$y = \pm \cos\gamma \sqrt{a^2 - x^2}.$$

$a \cos\gamma$ is a constant, and is evidently that line of the ellipse which replaces that radius of the circle which is at right angles to the $x$-axis. Put $a \cos\gamma = b$; then $\cos\gamma = \frac{b}{a}$, and the ellipse-equation becomes

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2};$$

that is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$a$ is called the **semi-major axis**, and $b$ the **semi-minor axis** of the ellipse.

The student should distinguish carefully between the axis of reference ($x$-axis) and the major axis. An axis of reference is a mere convenience, and we might study the ellipse with such an axis occupying some other position in relation to the curve, or even without any such axis; but the major axis is an essential line of the ellipse, occupying a special position within it. The same is true of the minor axis.

**110.** Now let the ellipse and the circle be drawn in the same plane, — the ellipse inside the circle with the major axis coinciding with that diameter of the circle which lies in the $x$-axis.

The ellipse has the appearance of a circle flattened in the direction $YY'$. $AA' = 2a$ is the major axis, and $BB' = 2b$ is the minor axis. $A$ and $A'$, the extremities of the major axis, are called the **vertices**.

With $B$, one extremity of the minor axis, as a center and with a radius equal to $a$, strike an arc. This arc will cut $AA'$ at points $F$ and $F'$ equally distant from $O$, the common center of the circle and the ellipse. Evidently the flatter the ellipse is, the farther these points will be from $O$; hence, if we know the ratio of $OF$ to $OA$, we know how flat the ellipse is, compared with the circumscribed circle.

$BF = a$, $BO = b$; therefore $OF = \sqrt{a^2 - b^2}$, and

$$\frac{OF}{OA} = \frac{\sqrt{a^2 - b^2}}{a}.$$

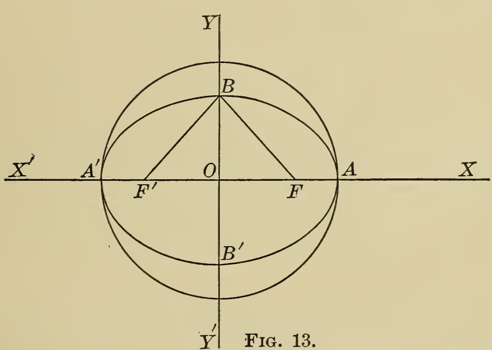


Fig. 13.

This important ratio is called the **eccentricity** of the ellipse, and is denoted by $e$. The somewhat similar ratio, $\frac{a - b}{a}$, is called the **ellipticity** of the ellipse.

The eccentricity is evidently a proper fraction. The points $F$ and $F'$ are called **foci**; and the double ordinate through either focus is known as the **latus rectum.**

**111.** Since $$OF = \sqrt{a^2 - b^2},$$

$$FA = a - \sqrt{a^2 - b^2} = a\left(1 - \frac{\sqrt{a^2 - b^2}}{a}\right) = a(1 - e).$$

Similarly, $$FA' = a(1 + e).$$

From the way in which the points $F$ and $F'$ were found we see that the sum of the distances $BF$ and $BF'$ is $2a$. It may now be shown that *the sum of the distances from any point* $P(x', y')$ *on the ellipse to the foci is* $2a$.

The coördinates of $F$, the right-hand focus, are $ae$, $0$; hence, by Art. 107,

$$(FP)^2 = (x' - ae)^2 + y'^2$$
$$= x'^2 - 2aex' + a^2e^2 + y'^2.$$

Since $P$ is on the ellipse, its coördinates must satisfy the equation to the ellipse, and we have

$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$$
$$= (1 - e^2)(a^2 - x'^2).$$

Substituting this value of $y'^2$ in the expression for $(FP)^2$,

$$(FP)^2 = x'^2 - 2aex' + a^2e^2 + (1 - e^2)(a^2 - x'^2)$$
$$= a^2 - 2aex' + e^2x'^2;$$

therefore $$FP = a - ex'.$$

Notice that the other root, $-(a - ex')$, is rejected because $a > ex'$ and $FP$ is positive. Repeating the argument for the distance $F'P$, the coördinates of $F'$ being $-ae$, 0, $F'P = a + ex'$; hence $FP + F'P = 2a$.

**112.** The proposition just established affords a way of mechanically constructing an ellipse. Fasten one end of a string at a point $F$ on the blackboard or paper, and the other at a point $F'$, taking the distance $FF'$ somewhat less than the length of the string. Pass the string around a pencil and move the point of the pencil over the paper, keeping the string taut. An ellipse will be described.

It is clear now that we might define an ellipse as *the path of a point which moves so that the sum of its distances from two fixed points is a constant.*

**113.** If $b = a$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation returns to the circle-equation $x^2 + y^2 = a^2$; also, $e = \frac{\sqrt{a^2 - a^2}}{a} = 0$. It thus appears that a circle is merely an ellipse with equal axes and eccentricity equal to zero.

**114.** Let a line $DD'$ (Fig. 14), be drawn parallel to the minor axis. If the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the minor axis lies in the $y$-axis, and hence the line $DD'$ is parallel to the $y$-axis. If its distance from the axis is $\frac{a}{e}$, its equation is $x = \frac{a}{e}$, the equation affirming that whatever may be the ordinate of the point tracing the line, the abscissa is constantly $\frac{a}{e}$. Now the distance from any point $P(x', y')$ on the curve to this line is $\frac{a}{e}$ minus the distance of the point from the $y$-axis; that is, $\frac{a}{e} - x'$, or $\frac{a - ex'}{e}$. We have already seen (Art. 111) that the distance from $P$ to the focus is $a - ex'$. Hence the distance from any point on the ellipse to the focus and the distance from the point to the line $x = \frac{a}{e}$ are in the ratio

$$\frac{a - ex'}{\frac{a - ex'}{e}};$$

that is, $e$. Accordingly the ellipse may be defined as *the path of a point which moves so that its distance from a given fixed point and its distance from a given fixed line have a constant ratio less than unity.*

The line $x = \frac{a}{e}$ is called the **directrix.**

**115.** In Fig. 14 let $FP = r$ and angle $EFP = \theta$.

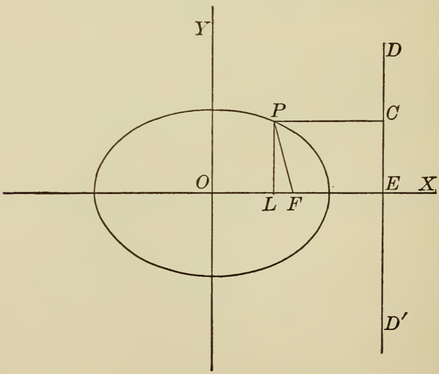


FIG. 14.

We have $$FE = \frac{a}{e} - ae;$$

$$FL = r\cos(180^\circ - \theta)$$

$$= -r\cos\theta,$$

and also $$\frac{FP}{PC} = e;$$

then $$FP = ePC = e(FE + FL);$$

that is, $$r = e\left(\frac{a}{e} - ae - r\cos\theta\right)$$

$$= a - ae^2 - er\cos\theta.$$

Solving for $r$, $$r = \frac{a(1-e^2)}{1 + e\cos\theta}.$$

This is the equation to the ellipse in polar coördinates with the pole at the right-hand focus. From the point of view of astronomy it is the most important of all forms of ellipse-equations. See Art. 90.

**116.** It is of interest, logically, to note that the theory of the ellipse may be developed from the definition given in Art. 112, or the one in Art. 114, or, indeed, from any fundamental property. In the present instance we have chosen to begin by viewing the ellipse as the *projection of a circle* on a plane making a given angle with the plane of the circle.

**117. The hyperbola.** About the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ circumscribe a rectangle with its sides parallel to the axes of the ellipse. Draw the diagonals of the rectangle. Half of one of the diagonals is $\sqrt{a^2 + b^2}$. With a radius of this length and with the center of the ellipse for center, strike an arc cutting the major axis produced in the points $F$ and $F'$. Now take $e = \frac{\sqrt{a^2 + b^2}}{a}$ and draw a line $DD'$ (Fig. 15) whose

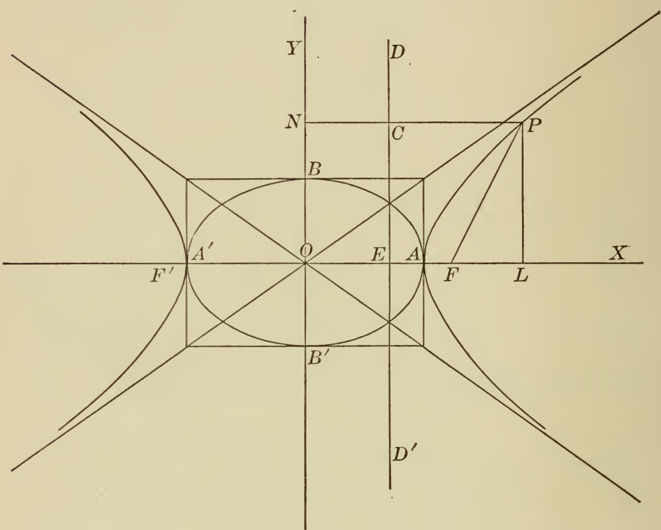


FIG. 15.

equation is $x = \frac{a}{e}$. This line will cut the ellipse, because $\frac{a}{e} < a$.

Following the analogy of the ellipse, we proceed to find the path of a point whose distance from the fixed point $F$ is to its distance from the fixed line $DD'$ in a constant ratio, — this ratio $e$ being here defined as $\frac{\sqrt{a^2+b^2}}{a}$, and hence greater than unity.

Let $P$ be the moving point whose coördinates are $x = PN$, $y = PL$. $PN$ cuts $DD'$ at $C$, and $DD'$ cuts the $x$-axis at $E$.

$$\frac{PF}{PC} = e; \text{ that is, } PF = ePC = e\left(x - \frac{a}{e}\right).$$

Also,
$$(PF)^2 = (PL)^2 + (FL)^2$$
$$= y^2 + (x - ae)^2.$$

Equating the two expressions for $(PF)^2$,

$$e^2\left(x - \frac{a}{e}\right)^2 = y^2 + (x - ae)^2.$$

Expanding and reducing,

$$x^2(e^2 - 1) - y^2 = a^2(e^2 - 1),$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1;$$

that is,
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

**118.** The curve whose equation we have now found is the **hyperbola.** Although closely related to the ellipse, it differs from that curve in various important respects:

1. $a$ and $b$ being the same in magnitude and position for the two curves, no portion of the hyperbola lies within the area occupied by the ellipse; for as soon as $x < a$, $y$ is imaginary.

2. The hyperbola has two parts or branches symmetrically placed with respect to the axial line in which $BB'$ lies; for if we assign values $-x'$, $-x''$, $-x'''$, etc., to $x$, we obtain the same values for $y$ that are obtained when $+x'$, $+x''$, $+x'''$, etc., are the values assigned.

3. Values indefinitely large may be assigned to $x$ without making $y$ imaginary. The curve, therefore, extends to infinity.

**119.** The equation to the hyperbola may be written

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2};$$

or, expanding $(x^2 - a^2)^{\frac{1}{2}}$ by the binomial theorem,

$$y = \pm \frac{b}{a}\left(x - \frac{a^2}{2x} - \frac{a^4}{8x^3} - \cdots\right). \qquad (a)$$

The equations to the two diagonals (produced) of the rectangle (Fig. 15) are seen to be

$$y = \pm \frac{b}{a}x. \qquad (b)$$

Comparing equations $(a)$ and $(b)$, we observe that any ordinate of the hyperbola is less than the corresponding ordinate of the lines; but we also notice that as $x$ becomes larger and larger, the ordinates, according to equations $(a)$ and $(b)$, approach equality.

Whenever such a relation exists between a line and a curve, — the distance between them becoming indefinitely small as the points describing them recede to infinity, — the straight line is called an **asymptote.**

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has therefore the asymptotes $y = \pm \frac{b}{a}x$.

**120.** The line $AA'$ (Fig. 15), $2a$ in length, is called the **transverse axis** of the hyperbola; and $BB'$, $2b$ in length, is the **conjugate axis.**

If we consider the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

we find it situated with respect to the $y$-axis just as the first hyperbola is with respect to the $x$-axis. This curve, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, is known as the **conjugate hyperbola** in distinction from the primary or **transverse hyperbola.**

Following the method of Art. 119, we find that the lines $y = \pm \frac{b}{a} x$ are asymptotes of the conjugate hyperbola also.

**121.** The points $A$ and $A'$ where the transverse axis meets the curve are its vertices. Similarly, $B$ and $B'$ are the vertices of the secondary or conjugate hyperbola.

The transverse and conjugate axes, the asymptotes, and the directrix are all essential lines of the hyperbola, and sustain a fixed geometric relation to it like a rigid framework, so that if the position of the hyperbola is changed with respect to the $x$- and $y$-axes, these lines go with it.

**122.** If $b = a$, the rectangle (Fig. 15) becomes a square; the asymptotes become $y = \pm x$, the two lines now crossing each other at right angles (Art. 104); the equation to the hyperbola itself becomes $x^2 - y^2 = a^2$, and is known as the **equilateral hyperbola.** It is evidently the hyperbola that would appear in plane $M$, Art. 109, in connection with the circle $x^2+y^2=a^2$ and the circumscribed square.

**123.** If we rotate the axes in the negative direction through the angle $-45°$, we shall have these axes coinciding with the asymptotes of the hyperbola $x^2-y^2=a^2$.

To do this we use the formulas

$$x = x\cos\alpha - y\sin\alpha,$$

$$y = x\sin\alpha + y\cos\alpha, \qquad \text{(Art. 101)}$$

which become, for $\alpha = -45°$,

$$x = x\tfrac{1}{2}\sqrt{2} + y\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\sqrt{2}(x+y),$$

$$y = -x\tfrac{1}{2}\sqrt{2} + y\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\sqrt{2}(y-x).$$

Substituting these values for $x$ and $y$ in the equation $x^2-y^2=a^2$, we have

$$[\tfrac{1}{2}\sqrt{2}(x+y)]^2 - [\tfrac{1}{2}\sqrt{2}(y-x)]^2 = a^2;$$

that is,

$$\tfrac{1}{2}(x+y)^2 - \tfrac{1}{2}(y-x)^2 = a^2,$$

which becomes, after reduction,

$$xy = \frac{a^2}{2}.$$

This is the equation to an equilateral hyperbola referred to its asymptotes.

The isotherm $pv=c$ (Art. 36) is an important illustration. (See Maxwell's *Theory of Heat*, Chap. VI.)

**124.** Following the method of Art. 111, and using Fig. 15, it is found that

$$(FP)^2 = e^2x'^2 - 2aex' + a^2.$$

Taking $FP$ positive, and noticing that $ex' > a$, we have

$$FP = ex' - a.$$

Similarly, $$F'P = ex' + a;$$

therefore $$F'P - FP = 2a,$$

and the hyperbola may be defined as *the path of a point moving so that the difference of its distances from two fixed points is a constant.*

**125.** The polar equation to the hyperbola may be readily obtained from Fig. 15.

Let $FP = r$, $F'P = r'$, and the angle $LFP = \theta$. We have also $r' - r = 2a$ and $F'F = 2ae$.

Then $$r'^2 = r^2 + 4ar + 4a^2,$$

and since $r'$ is one side of the triangle $F'PF$,

$$r'^2 = r^2 + 4a^2e^2 - 4aer\cos(180° - \theta).$$

Equating these two values of $r'^2$,

$$r^2 + 4ar + 4a^2 = r^2 + 4a^2e^2 + 4aer\cos\theta;$$

hence $$r = \frac{a(e^2 - 1)}{1 - e\cos\theta}.$$

This equation may be described as the right-hand focal polar equation to the hyperbola.

When $\theta = 0$, $r = -a(1 + e)$, and the feather-end of the arrow (Art. 45) gives the vertex of the left-hand branch of the curve. As the radius vector continues to revolve in the positive direction, we continue to get negative values of $r$ and points on the left-hand branch until $\theta = \cos^{-1}\frac{1}{e}$; $r$ is then infinite, and the radius vector is parallel to the asymptote $y = \frac{b}{a}x$, because $\cos^{-1}\frac{1}{e} = \tan^{-1}\frac{b}{a}$.

From $\theta = \cos^{-1}\frac{1}{e}$ to $\theta = 360° - \cos^{-1}\frac{1}{e}$, $r$ is positive, and the right-hand branch is being traced. Finally, when $\theta$ changes from $360° - \cos^{-1}\frac{1}{e}$ to $360°$, the remaining part of the left-hand branch is traced.

**126. The parabola.** It remains to inquire what kind of a curve we have when a point moves so that its distance from a fixed point is to its distance from a fixed line in the constant ratio unity.

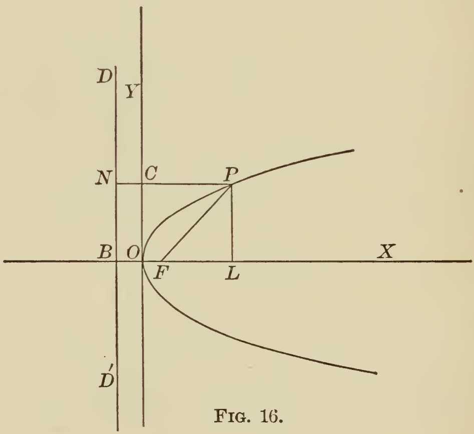


Fig. 16.

Let $DD'$, Fig. 16, be the fixed line, and $F$ the fixed point. Let $2p$ be the length of $FB$, the distance from $F$ to $DD'$. Take a line through $F$ perpendicular to

$DD'$ for the $x$-axis, and a line parallel to $DD'$, bisecting $FB$, for the $y$-axis. Let $P$ be the moving point with the coördinates $OL$ and $PL$.

Then, $$FP = PN = x + p;$$

also, $$(FP)^2 = (PL)^2 + (FL)^2$$
$$= y^2 + (x - p)^2.$$

Hence, $$(x + p)^2 = y^2 + (x - p)^2;$$

that is, $$y^2 = 4\,px.$$

This curve is called the **parabola.**

Since it is the path of a point whose distance from the fixed point is to its distance from the fixed line in the ratio unity, it must be regarded as the transition curve between the ellipse and the hyperbola. Its eccentricity $e$, being the ratio of the two distances, is of course unity.

**127.** Considering the equation $y^2 = 4\,px$, we see that the parabola has a *line of symmetry* which has been used as the $x$-axis; for if any value be assigned to $x$, $y$ has two values numerically equal and with opposite signs; so that if the area above the $x$-axis were folded over, the part of the curve in the upper area would exactly fit the part in the lower. This line of symmetry is called the *axis* of the curve, and the point where it meets the curve is the *vertex*. The point $F$ is the *focus*, and the double ordinate through the focus is the *latus rectum*, as in the case of the ellipse.

When $x = p$, $y = 2\,p$; therefore the semi-latus rectum is twice the distance of the focus from the vertex.

**128.** If $p$ is positive in the equation $y^2 = 4px$, positive values may be assigned to $x$ without making $y$ imaginary. Hence, the parabola like the hyperbola extends to infinity. It differs from the hyperbola, however, in this important respect: it has only one real branch, because negative values of $x$ make $y$ imaginary.

**129.** If $p$ is negative in the equation $y^2 = 4px$, we have the same law as before, governing the motion of the point $P$; but the path is now wholly on the negative side of the $y$-axis, for only negative values can now be assigned to $x$.

Similarly, $x^2 = 2py$ is a parabola above the $x$-axis, with the $y$-axis for its line of symmetry, if $p$ is positive; while $x^2 = 2py$ is the same curve below the $x$-axis if $p$ is negative.

**130.** The polar equation to the parabola, the focus being pole, is obtained from Fig. 16. $FP$ is $r$, and the angle $XFP$ is $\theta$. Then

$$FL = r\cos\theta,$$

but $$r = PN = FL + 2p;$$

therefore, $$r = r\cos\theta + 2p;$$

that is, $$r = \frac{2p}{1-\cos\theta}.$$

**131.** The ellipse, parabola, and hyperbola are known as *conic sections;* for it can be shown that if a cone of revolution is cut by a plane in any manner whatever, the cross-section is one of these three curves. See

Puckle's *Conic Sections*, Arts. 323–325, together with Chap. VIII of that treatise.

**132.** A straight line becomes a tangent to a curve at any point $(x', y')$ if (1) it passes through that point, and (2) if it has the same slope as the curve at that point. We have already seen (Art. 38) that if $y = f(x)$ is the equation to a curve, $\frac{dy}{dx}$ gives its slope or gradient at each point. It has also been shown (Art. 107) that

$$y - y' = \frac{dy}{dx}(x - x')$$

is the equation to a straight line passing through the point $(x', y')$ with the slope $\frac{dy}{dx}$. It follows that if $(x', y')$ is a point $P'$ on the curve $y = f(x)$ and if $\frac{dy}{dx}$ is specialized for that point, becoming $\left.\frac{dy}{dx}\right]_{x=x'}$ or $\frac{dy'}{dx'}$,

$$y - y' = \frac{dy'}{dx'}(x - x')$$

is the equation to the tangent at $P'$.

For example, let us find the equation to the tangent at the upper extremity of the latus rectum of the parabola $y^2 = 4px$. Differentiating, we have $\frac{dy}{dx} = \frac{2p}{y}$. The coördinates of the upper extremity of the latus rectum are $p$, $2p$. Specializing $\frac{dy}{dx}$ for this point.

$$\left.\frac{dy}{dx}\right]_{y=2p} = \frac{2p}{2p} = 1;$$

and the general equation gives

$$y - 2p = x - p;$$

that is, $$y = x + p,$$

which is the equation to the tangent in question. We notice that this particular tangent makes an angle of 45° with the axis of the curve, which is here the $x$-axis, and cuts the axis produced where the directrix $DD'$ cuts it. (See Fig. 16.)

*Example.* Find the general equation to the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Differentiating and solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y},$$

and the general equation to the tangent becomes, for the ellipse,

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x').$$

For instance, the coördinates of the upper extremity of the left-hand latus rectum are $-ae$, $b\sqrt{1-e^2}$; $\frac{dy}{dx}$ therefore becomes

$$\frac{b^2}{a^2}\frac{ae}{b\sqrt{1-e^2}}; \text{ that is, } e;$$

and the tangent at the point named is

$$y - b\sqrt{1-e^2} = e(x + ae).$$

It will be noticed that this particular tangent makes with the major axis an angle whose tangent is the eccentricity.

The line perpendicular to the tangent at the point of tangency is called the **normal**. Its equation is evidently

$$y - y' = -\frac{dx'}{dy'}(x - x').$$

**133.** The general equation to the tangent to the parabola $y^2 = 4px$, in terms of the coördinates of the point of tangency, is seen to be

$$y - y' = \frac{2p}{y'}(x - x');$$

that is, $$yy' = 2px + y'^2 - 2px';$$

or, since $y'^2 = 4px'$, $(x', y')$ being on the parabola,

$$yy' = 2p(x + x'), \qquad (1)$$

and $$y = \frac{2p}{y'}(x + x'). \qquad (2)$$

Writing the equation to the line passing through the focus and $P(x', y')$, we have, after reducing,

$$y = \frac{y'}{x' - p}(x - p). \qquad (3)$$

If $A$ is the point where the tangent cuts the axis produced, $FPA$ is an isosceles triangle. To prove this, let $\tan\alpha = \frac{2p}{y'}$, the coefficient of $x$ in (2). Then

$$\tan 2\alpha = \frac{\frac{4p}{y'}}{1 - \frac{4p^2}{y'^2}} = \frac{4py'}{y'^2 - 4p^2} = \frac{4py'}{4px' - 4p^2} = \frac{y'}{x' - p}.$$

But this is the coefficient of $x$ in (3).

Hence, $\quad$ angle $PAF =$ angle $FPA$.

It immediately follows that if a line $PL$ is drawn on the concave side of the parabola, parallel to its axis, $FP$ and $PL$ make equal angles with the tangent.*

Advantage is taken of this property of the parabola in the construction of reflectors. Since the angle of reflection of a ray of heat or light equals the angle of incidence, if a light be placed at the focus of a parabolic reflector, the light is reflected in a system of (approximately) parallel rays. Illumination of a railroad track for a long distance in front of the locomotive is secured by means of such a reflector. Conversely, if rays of heat or light, parallel to the axis of a parabolic reflector, fall upon its concave surface, they will converge at the focus.

**134.** It is required to find the locus (path) of the middle point of any ellipse-chord moving parallel to itself.

Let $C(x'', y'')$ and $C'(x', y')$ be the points where the chord meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let $M(x, y)$ bisect the chord $CC'$.

* "Hertz, in the first of his celebrated experiments on the propagation of electric rays, made use of this property of parabolic surfaces. He employed large reflectors of sheet zinc bent into the form of parabolic cylinders, in whose focal line the transmitter and the receiver of the electric waves were placed. The electric rays passed from the transmitter to the first parabolic reflector, were there reflected so as to become parallel, and were then reflected from the second reflector to the receiver placed at its focus." — Young and Linebarger's *Calculus*.

Then $x = \frac{1}{2}(x' + x'')$, and $y = \frac{1}{2}(y' + y'')$;

that is, $x' = 2x - x''$, and $y' = 2y - y''$.

Since $(x', y')$ and $(x'', y'')$ are each on the ellipse,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \qquad (1)$$

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1. \qquad (2)$$

Substituting in (1) the values just noticed for $x'$ and $y'$, we have

$$\frac{(2x - x'')^2}{a^2} + \frac{(2y - y'')^2}{b^2} = 1,$$

or
$$\frac{4x^2 - 4xx''}{a^2} + \frac{x''^2}{a^2} + \frac{4y^2 - 4yy''}{b^2} + \frac{y''^2}{b^2} = 1.$$

Introducing relation (2),

$$\frac{x(x - x'')}{a^2} + \frac{y(y - y'')}{b^2} = 0,$$

and therefore
$$\frac{y - y''}{x - x''} = -\frac{b^2}{a^2}\frac{x}{y}.$$

Now let
$$y - y'' = m(x - x'')$$

be the equation to the chord $CC'$; then

$$\frac{y - y''}{x - x''} = m,$$

and this is true, of course, when $(x, y)$, the point tracing the line $CC'$, is restricted to the point $M$.

Equating the two values of $\dfrac{y - y''}{x - x''}$,

$$-\frac{b^2}{a^2}\frac{x}{y} = m;$$

that is,
$$y = -\frac{b^2}{a^2 m}x.$$

The path of the middle point of any chord of slope $m$, moving parallel to itself across the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is therefore a straight line passing through the center of the ellipse.

If $m = 0$, the equation becomes $x = 0$, the equation to the minor axis; and if $m = \infty$, $y = 0$, the equation to the major axis.

Writing $-b^2$ for $b^2$, we have

$$y = \frac{b^2}{a^2 m}x,$$

the corresponding equation in relation to the hyperbola.

**135.** By means of the result in the preceding article, we are now able to find the center and construct the axes of any ellipse.

Draw any two parallel chords and bisect them. The chord passing through the points of bisection must pass through the center of ellipse; and the point of bisection of this third chord is the center. With the center now found and a radius of any convenient length, strike a circle cutting the ellipse. Draw one chord common to both ellipse and circle, and finally draw an ellipse-chord perpendicular to the preceding chord at its middle point. It will be the major (or minor) axis of the ellipse.

Having given the hyperbola with its accompanying conjugate hyperbola, the construction of the axes is the same as for the ellipse.

**136.** The tangent-equation, Art. 132, involves the coördinates of the point of tangency. It is desirable to obtain a form in which these coördinates do not appear. What conditions must be imposed on the line $y = mx + n$ so that it shall keep the slope $m$ and yet be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

Eliminating $y$ between the equations

$$y = mx + n, \tag{1}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2}$$

the resulting equation,

$$\frac{x^2}{a^2} + \frac{(mx+n)^2}{b^2} = 1,$$

has for its roots the abscissas of the points of intersection of (1) and (2). These roots are

$$x = -\frac{\frac{mn}{b^2}}{\frac{1}{a^2} + \frac{m^2}{b^2}} \pm \sqrt{\frac{\frac{m^2n^2}{b^4}}{\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)^2} - \frac{\frac{n^2 - b^2}{b^2}}{\frac{1}{a^2} + \frac{m^2}{b^2}}}.$$

Thus far the straight line is merely a secant (real or imaginary) of the ellipse. If it is to become a tangent, the two points of its intersection with the ellipse must be indefinitely near to each other; that is, the two abscissas must be equal. Hence, the radical, which now makes them unequal, must vanish, and we have

$$\frac{\frac{m^2n^2}{b^4}}{\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)^2}-\frac{\frac{n^2-b^2}{b^2}}{\frac{1}{a^2}+\frac{m^2}{b^2}}=0.$$

From this equation of condition we obtain

$$n=\pm\sqrt{a^2m^2+b^2},$$

which is therefore the relation which must hold between $n$ and $a$, $m$ and $b$ in order that (1) shall be tangent to (2), and we have

$$y=mx\pm\sqrt{a^2m^2+b^2}. \tag{3}$$

The double sign in (3) plainly means two tangents parallel to each other, one cutting the $y$-axis at the distance $\sqrt{a^2m^2+b^2}$ above the origin, and the other at the same distance below it.

The corresponding equation for the tangent to the hyperbola may be obtained at once by writing $-b^2$ for $b^2$ in (3), and we have

$$y=mx\pm\sqrt{a^2m^2-b^2}. \tag{4}$$

If $b=a$, so that the ellipse becomes a circle, (3) becomes

$$y=mx\pm a\sqrt{m^2+1}. \tag{5}$$

Similarly, if the hyperbola is equilateral, (4) becomes

$$y=mx\pm a\sqrt{m^2-1}. \tag{6}$$

### Exercises.

**137.** 1. Construct the ellipse

$$\frac{x^2}{4}+\frac{y^2}{9}=1.$$

What is its eccentricity? How must the formula for $e$ be written in this case?

2. Find the points of intersection of the ellipse and hyperbola whose equations are

$$x^2+2y^2=1,\quad 3x^2-6y^2=1,$$

and show that at each of these points the tangent to the ellipse is the normal to the hyperbola. (Puckle's *Conic Sections.*)

3. Find the equations to the asymptotes of the hyperbola $3x^2-6y^2=1$.

4. Find the distance between the right-hand focus of $x^2+2y^2=1$ and the right-hand focus of $3x^2-6y^2=1$.

5. Write the equation to a circle which shall have its center coincident with the focus of the parabola $y^2=4px$, and shall be tangent to the parabola.

6. Find an expression for the perpendicular distance from the right-hand focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent $y=mx+\sqrt{a^2m^2+b^2}$.

7. Find an expression for the perpendicular distance from the focus of the parabola $y^2=4px$ to any normal.

8. Show that the line $y=mx+n$ becomes a tangent to the parabola $y^2=4px$ if $n=\frac{p}{m}$.

9. Show that the path of the middle point of any parabola-chord moving parallel to itself is a line parallel to the axis of the parabola.

**10.** Given a parabola, find its axis and focus.

**11.** The locus of the foot of the perpendicular from the center of the equilateral hyperbola $x^2 - y^2 = a^2$ is the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Use $$y = mx + a\sqrt{m^2 - 1}.$$

The line perpendicular to it passing through the center is $y = -\frac{1}{m}x$. It is required to find the path of the intersection of these two lines as $m$ passes through all values. Eliminate $m$.

**12.** Show that the locus of the foot of the perpendicular dropped from the focus of the parabola on its tangent is the tangent at the vertex.

**13.** If a source of light or heat is placed in one focus of an ellipse, the rays will be reflected so as to meet in the other focus.

**14.** A planet at $P$ is moving in the direction $PQ$. Its distance $PS$ from the sun at $S$ (one focus of its elliptic orbit) is $\frac{1}{4}$ its major axis. Construct the orbit.

**15.** Given one focus and any point $P$ and the length of the major axis of an ellipse; show that the eccentricity depends on the direction of the tangent at $P$. Construct the major and minor axes of ellipses corresponding to various tangents through $P$.

**16.** The tangent at any point of a hyperbola is produced to meet the asymptotes; show that the triangle cut off is of constant area.

**17.** Find the equation to the path of the center of a circle which is tangent to two given circles.

**138.** Just as a point in a plane may be determined by referring it to two lines at right angles to each other, so a point in space may be determined by referring it to three *planes*, each plane intersecting the other two at right angles. The point common to the three planes

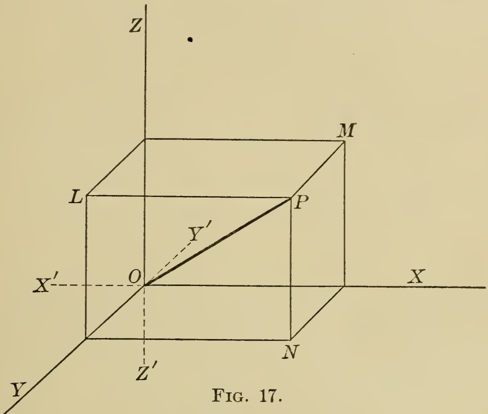


FIG. 17.

is called the origin, and the lines of intersection of the planes are known as the axes of $x$, $y$, and $z$. The positive directions of the axes are usually taken to be represented in the figure by $OX$, $OY$, and $OZ$; the negative directions are then $OX'$, $OY'$, $OZ'$.

If $P$ (Fig. 17) is any point in space, then $PL$, $PM$, $PN$, its perpendicular distances from the planes $YOZ$, $ZOX$, and $XOY$ respectively are the coördinates $x$, $y$, and $z$ respectively.

**139.** The three planes evidently divide the space around the origin into eight equal triedral angles. If a point is in the upper front right-hand angle, its coördinates are all positive, because each one is measured parallel to its own axis and in the positive direction. Again, if a point is in the upper front left-hand angle, the $y$ and $z$ coördinates are positive, but the $x$ coördinate is negative because measured in the negative direction parallel to $OX'$. In like manner we are able to state the character of each coördinate for points situated in each one of the other six angles.

**140.** $OP$, the distance of $P$ from the origin, is the diagonal of the rectangular parallelopiped, three of whose edges are $PL$, $PM$, $PN$. Therefore,

$$\begin{aligned}(OP)^2 &= (PL)^2 + (PM)^2 + (PN)^2 \\ &= x^2 + y^2 + z^2.\end{aligned}$$

**141.** Suppose we have any two points $P'(x', y', z')$ and $P''(x'', y'', z'')$. Let planes be passed through $P'$ and $P''$ parallel to the three planes of reference. There is thus formed a rectangular parallelopiped whose diagonal is the line $P'P''$, and three of whose edges are $x'' - x'$, $y'' - y'$, $z'' - z'$. Therefore,

$$(P'P'')^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2.$$

This formula for the distance between two points in space should be compared with the formula in Art. 107.

**142.** We have seen (Art. 39) that the general equation of the first degree in two variables represents a straight line. It may now be asked, What is represented by

$$Ax + By + Cz + D = 0, \qquad (1)$$

the general equation of the first degree in three variables?

1. It represents a *surface* and not a solid. For let $(a, b)$ be a point in the plane $YOX$, and suppose that a straight line be drawn through this point parallel to the $z$-axis to meet the locus of equation (1), whatever kind of locus it may be. We now have

$$z = \frac{-Aa - Bb - D}{C};$$

therefore the straight line meets the locus in one definite point at the distance $\frac{-Aa - Bb - D}{C}$ from the plane $YOX$, and consequently the locus cannot be made up of layers either adjacent to one another or occurring at intervals.

2. The surface is a *plane*. For suppose that a point moving in the surface be so restricted that it must remain at a constant distance from the plane $YOX$; that is, let $z$ have a constant value, say $c$. Equation (1) is now reduced to $Ax + By + Cc + D = 0$. Therefore the point moving in the surface and at a constant distance from the plane $YOX$ is moving in a straight line. In other words, any section of the surface made by a plane parallel to the plane $YOX$ is a straight line. Hence, if $Ax + By + Cz + D = 0$ is not a *plane* surface, it must be a wavy surface, something like a corrugated tin roof with the corrugation lines parallel to the plane $YOX$. But repeating the argument, making $y$ a constant, we find that all sections made by planes parallel to the plane $ZOX$ are straight lines. The surface in question must therefore be a plane.

In case the constant term $D$ is zero, the coördinates of the origin (0, 0, 0) satisfy the equation, and the plane passes through the origin.

Thus the equation, $Cz - C'y + C''x = 0$ (p. 103), represents a plane passing through the origin; and any moving point whose coördinates satisfy this equation at each instant, must be moving in the plane and hence in a plane curve.

Arts. 138–142 have been introduced for the sake of Arts. 86–88.

# CHAPTER V.

## FORMULAS.

1. $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}.$ $1_1.$ $\int\left(\frac{du}{dx}dx+\frac{dv}{dx}dx\right)=u+v.$

2. $\frac{d}{dx}uv=v\frac{du}{dx}+u\frac{dv}{dx}.$ $2_1.$ $\int\left(v\frac{du}{dx}dx+u\frac{dv}{dx}dx\right)=uv.$

3. $\frac{d}{dx}u^n=nu^{n-1}\frac{du}{dx}.$ $3_1.$ $\int u^m\frac{du}{dx}dx=\frac{u^{m+1}}{m+1}.$

4. $\frac{d}{dx}\sin u=\cos u\frac{du}{dx}.$ $4_1.$ $\int\cos u\frac{du}{dx}dx=\sin u.$

5. $\frac{d}{dx}\cos u=-\sin u\frac{du}{dx}.$ $5_1.$ $\int\sin u\frac{du}{dx}dx=-\cos u.$

6. $\frac{d}{dx}\tan u=\sec^2 u\frac{du}{dx}.$ $6_1.$ $\int\sec^2 u\frac{du}{dx}dx=\tan u.$

7. $\frac{d}{dx}\cot u=-\operatorname{cosec}^2 u\frac{du}{dx}.$ $7_1.$ $\int\operatorname{cosec}^2 u\frac{du}{dx}dx=-\cot u.$

8. $\frac{d}{dx}\sec u=\tan u\sec u\frac{du}{dx}.$ $8_1.$ $\int\tan u\sec u\frac{du}{dx}dx=\sec u.$

9. $\frac{d}{dx}\operatorname{cosec} u=-\cot u\operatorname{cosec} u\frac{du}{dx}.$

$9_1.$ $\int\cot u\operatorname{cosec} u\frac{du}{dx}dx=-\operatorname{cosec} u.$

10. $\frac{d}{dx}\sin^{-1}u=\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$ $10_1.$ $\int\frac{\frac{du}{dx}}{\sqrt{1-u^2}}dx=\sin^{-1}u.$

11. $\frac{d}{dx}\cos^{-1}u=-\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$ $11_1.$ $\int-\frac{\frac{du}{dx}}{\sqrt{1-u^2}}dx=\cos^{-1}u.$

12. $\frac{d}{dx}\tan^{-1}u = \frac{\frac{du}{dx}}{1+u^2}.$ $12_1.$ $\int \frac{\frac{du}{dx}}{1+u^2}\,dx = \tan^{-1}u.$

13. $\frac{d}{dx}\cot^{-1}u = -\frac{\frac{du}{dx}}{1+u^2}.$ $13_1.$ $\int -\frac{\frac{du}{dx}}{1+u^2}\,dx = \cot^{-1}u.$

14. $\frac{d}{dx}\sec^{-1}u = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$ $14_1.$ $\int \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}\,dx = \sec^{-1}u.$

15. $\frac{d}{dx}\operatorname{cosec}^{-1}u = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$

$15_1.$ $\int -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}\,dx = \operatorname{cosec}^{-1}u.$

16. $\frac{d}{dx}e^x = e^x.$ $16_1.$ $\int e^x dx = e^x.$

17. $\frac{d}{dx}a^x = a^x \log_e a.$ $17_1.$ $\int a^x \log_e a\,dx = a^x.$

18. $\frac{d}{dx}e^u = e^u\frac{du}{dx}.$ $18_1.$ $\int e^u\frac{du}{dx}\,dx = e^u.$

19. $\frac{d}{dx}a^u = a^u\frac{du}{dx}\log_e a.$ $19_1.$ $\int a^u\frac{du}{dx}\,dx = \frac{a^u}{\log_e a}.$

20. $\frac{d}{dx}\log_e x = \frac{1}{x}.$ $20_1.$ $\int\frac{dx}{x} = \log_e x.$

21. $\frac{d}{dx}\log_e u = \frac{\frac{du}{dx}}{u}.$ $21_1.$ $\int\frac{\frac{du}{dx}}{u}\,dx = \log_e u.$

22. $\frac{d}{dx}\sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}}.$ $22_1.$ $\int\frac{\frac{du}{dx}}{2\sqrt{u}}\,dx = \sqrt{u}.$

23. $\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2}.$

24. $\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2} [x\sqrt{x^2 \pm a^2} \pm a^2 \log (x + \sqrt{x^2 \pm a^2})].$

25. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}).$

26. $\int \frac{x}{\sqrt{a^2 \pm x^2}}\, dx = \pm \sqrt{a^2 \pm x^2}.$

27. $\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$

28. $\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\, dx = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).$

29. $\int x^2 \sqrt{a^2 - x^2}\, dx = \frac{x}{8} (2\,x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1} \frac{x}{a}.$

30. $\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a} \log \frac{x}{a + \sqrt{a^2 \pm x^2}}.$

31. $\int \frac{dx}{x^2\sqrt{a^2 \pm x^2}} = -\frac{\sqrt{a^2 \pm x^2}}{a^2 x}.$

32. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\, dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a}.$

33. $\int \frac{x}{\sqrt{2\,ax - x^2}}\, dx = a \operatorname{vers}^{-1} \frac{x}{a} - \sqrt{2\,ax - x^2}.$

34. $\int \frac{dx}{x\sqrt{2\,ax - x^2}} = -\frac{\sqrt{2\,ax - x^2}}{ax}.$

35. $\int \sqrt{2\,ax - x^2}\, dx = \frac{a^2}{2} \operatorname{vers}^{-1} \frac{x}{a} + \frac{x - a}{2} \sqrt{2\,ax - x^2}.$

36. $\int \frac{\sqrt{2\,ax - x^2}}{x}\, dx = a \operatorname{vers}^{-1} \frac{x}{a} + \sqrt{2\,ax - x^2}.$

37. $\int \tan x \, dx = -\log \cos x.$ 38. $\int \cot x \, dx = \log \sin x.$

39. $\int \frac{dx}{\sin x} = \log \tan \frac{x}{2}.$ 40. $\int \frac{dx}{\cos x} = \log \tan \left( \frac{x}{2} + \frac{\pi}{4} \right).$

41. $\int \frac{dx}{\sin^2 x} = -\cot x.$ 42. $\int \frac{dx}{\cos^2 x} = \tan x.$

43. $\int \sin x \cos x \, dx = -\frac{1}{4} \cos 2x = \frac{\sin^2 x}{2}.$

44. $\int \frac{dx}{\sin x \cos x} = \log \tan x.$

45. $\int x \sin x \, dx = \sin x - x \cos x.$

46. $\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x.$

47. $\int \sin^2 x \, dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x.$

48. $\int \cos^2 x \, dx = \frac{1}{4} \sin 2x + \frac{1}{2} x.$

49. $\int \log x \, dx = x \log x - x.$

50. $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

51. $\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \log (1 + x^2).$

52. $\int \sec^{-1} x \, dx = x \sec^{-1} x - \log (x + \sqrt{x^2 + 1}).$

53. If $a > b$, $\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right).$

For other integrals see Peirce's *Short Table of Integrals.*

54. $f(z+x)=f(z)+f'(z)x+\dfrac{f''(z)x^2}{\underline{|2}}+\dfrac{f'''(z)x^3}{\underline{|3}}+\cdots.$

55. $f(x)=f(o)+f'(o)x+\dfrac{f''(o)x^2}{\underline{|2}}+\dfrac{f'''(o)x^3}{\underline{|3}}+\cdots.$

56. $(a+x)^m=a^m+ma^{m-1}x+\dfrac{m(m-1)a^{m-2}x^2}{\underline{|2}}$

$$+\frac{m(m-1)(m-2)a^{m-3}x^3}{\underline{|3}}+\cdots.$$

57. $e^x=1+x+\dfrac{x^2}{\underline{|2}}+\dfrac{x^3}{\underline{|3}}+\cdots.$

58. $a^x=1+x\log_e a+\dfrac{x^2(\log_e a)^2}{\underline{|2}}+\dfrac{x^3(\log_e a)^3}{\underline{|3}}+\cdots.$

59. $\log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots.$

60. $\sin x=x-\dfrac{x^3}{\underline{|3}}+\dfrac{x^5}{\underline{|5}}-\cdots.$ 61. $\cos x=1-\dfrac{x^2}{\underline{|2}}+\dfrac{x^4}{\underline{|4}}-\cdots$

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62. $\sin\left(\dfrac{\pi}{2}+\alpha\right)=\cos\alpha.$ 63. $\cos\left(\dfrac{\pi}{2}+\alpha\right)=-\sin\alpha.$

64. $\tan\left(\dfrac{\pi}{2}+\alpha\right)=-\cot\alpha.$ 65. $\cot\left(\dfrac{\pi}{2}+\alpha\right)=-\tan\alpha.$

66. $\sin(-\alpha)=-\sin\alpha;\ \cos(-\alpha)=\cos\alpha.$

67. $\tan(-\alpha)=-\tan\alpha;\ \cot(-\alpha)=-\cot\alpha.$

68. $\sin^2\alpha+\cos^2\alpha=1.$ 69. $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}.$

70. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$

71. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$

72. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$

73. $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$

74. $\sin 2\alpha=2\sin\alpha\cos\alpha.$

75. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$

76. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$

77. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

78. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}.$

79. $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha}.$

80. $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{2}}.$

81. $\cos\dfrac{\alpha}{2} = \sqrt{\dfrac{1+\cos\alpha}{2}}.$

82. $\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}.$

83. $\cot\dfrac{\alpha}{2} = \sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}.$

84. $\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta).$

85. $\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$

86. $\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta).$

87. $\cos\alpha - \cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$

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88. $\log ab = \log a + \log b.$

89. $\log\dfrac{a}{b} = \log a - \log b.$

90. $\log a^n = n\log a.$

91. $\log a^{\frac{1}{n}} = \dfrac{1}{n}\log a.$

92. If $ax^2 + bx + c = 0$,

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

93. $\underline{|n} = 1\cdot 2\cdot 3\cdot 4 \cdots n.$

94. $\log 1 = 0.$

95. $\log 0 = -\infty.$

96. $\log_a a = 1.$

97. $e = 2.7182818284 \cdots.$

98. $\log_{10} e = 0.43429448 \cdots.$

99. $\pi = 3.14159265 \cdots.$

100. $\log_{10}\pi = 0.49714987 \cdots.$

101. $R^\circ = \dfrac{180^\circ}{\pi} = 57^\circ.2957795 \cdots.$

102. $R'' = \dfrac{180 \cdot 60 \cdot 60''}{\pi} = 206264''.8 \cdots.$

103. $\log R^\circ = 1.75812263 \cdots.$

104. $\log R'' = 5.31442513 \cdots.$

105.

| | 30° | 45° | 60° |
|---|---|---|---|
| sin, | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{3}$ |
| cos, | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}$ |

106. Base of right triangle $= h \cos \gamma$; alt. $= h \sin \gamma$.
($h$ = hypothenuse; $\gamma$ = angle at base.)

107. Area of sector of circle $= \frac{1}{2} r (r\theta) = \frac{1}{2} r^2\theta$.

108. Area of ellipse $= \pi ab$.

# INDEX.

The references are to pages.

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